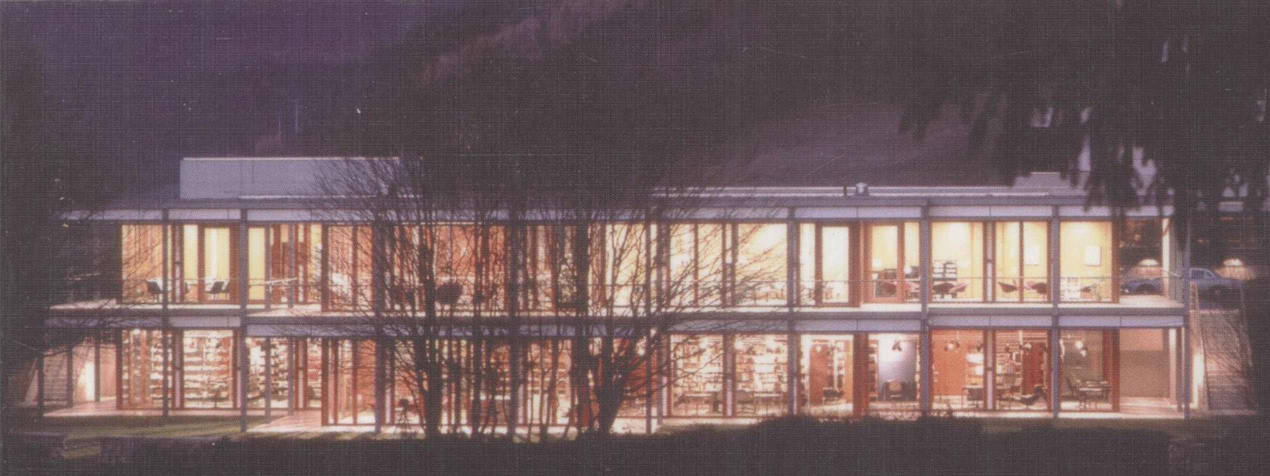

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The Novikov Conjecture

Geometry and Algebra

Matthias Kreck
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Authors:

Matthias Kreck
Mathematisches Institut
Ruprecht-Karls-Universität Heidelberg
Im Neuenheimer Feld 288
69120 Heidelberg
Germany
e-mail: kreck@mathi.uni-heidelberg.de

Wolfgang Lück
Mathematisches Institut
Westfälische Wilhelms-Universität Münster
Einsteinstrasse 62
48149 Münster
Germany
e-mail: lueck@math.uni-muenster.de

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Introduction

Manifolds are the central geometric objects in modern mathematics. An attempt to understand the nature of manifolds leads to many interesting questions. One of the most obvious questions is the following.

Let M and N be manifolds: how can we decide whether M and N are homotopy equivalent or homeomorphic or diffeomorphic (if the manifolds are smooth)?

The prototype of a beautiful answer is given by the Poincaré Conjecture. If N is S^n, the n-dimensional sphere, and M is an arbitrary closed manifold, then it is easy to decide whether M is homotopy equivalent to S^n. This is the case if and only if M is simply connected (assuming $n > 1$, the case $n = 1$ is trivial since every closed connected 1-dimensional manifold is diffeomorphic to S^1) and has the homology of S^n. The *Poincaré Conjecture* states that this is also sufficient for the existence of a homeomorphism from M to S^n. For $n = 2$ this follows from the well-known classification of surfaces. For $n > 4$ this was proved by Smale and Newman in the 1960s, Freedman solved the case in $n = 4$ in 1982 and recently Perelman announced a proof for $n = 3$, but this proof has still to be checked thoroughly by the experts. In the smooth category it is not true that manifolds homotopy equivalent to S^n are diffeomorphic. The first examples were published by Milnor in 1956 and together with Kervaire he analyzed the situation systematically in the 1960s.

For spheres one only needs very little information to determine the homeomorphism type: the vanishing of the fundamental group and control of the homology groups. Another natural class of manifolds is given by aspherical manifolds. A CW-complex is called *aspherical* if the homotopy groups vanish in dimension > 1, or, equivalently, if its universal covering is contractible. The *Borel Conjecture*, which is closely related to the Novikov Conjecture, implies that the fundamental group determines the homeomorphism type of an aspherical closed manifold.

For more general manifolds with prescribed fundamental group the classification is in general unknown even if the fundamental group is trivial. In this situation it is natural to construct as many invariants as possible hoping that at least for certain particularly important classes of manifolds one can classify them in terms of theses invariants. The most important invariants after homotopy and (co)homology groups are certainly characteristic classes which were defined and systematically treated in the 1950s. There are two types of characteristic classes for smooth manifolds: the *Stiefel–Whitney classes* $w_k(M)$ in $H^k(M;\mathbb{Z}/2)$ and the *Pontrjagin classes* $p_k(M) \in H^{4k}(M;\mathbb{Z})$. The nature of these classes is rather different. The Stiefel–Whitney classes of a closed manifold can be expressed in terms of cohomology operations and so are homotopy invariants, the Pontrjagin classes are diffeomorphism invariants (for smooth manifolds, and only for those they are a priori defined), but not homeomorphism or even homotopy invariants in general. Only very special linear combinations of the Pontrjagin classes are actually homotopy invariants.

For example, the first Pontrjagin class of a closed oriented 4-manifold $p_1(M)$ is a homotopy invariant. The reason is that $\langle p_1(M), [M]\rangle = 3\cdot \mathrm{sign}(M)$, where $\mathrm{sign}(M)$ is the *signature* of the intersection form on $H^2(M;\mathbb{Q})$. The signature is by construction a homotopy invariant. More generally, Hirzebruch defined a certain rational polynomial in the Pontrjagin classes (for a definition of Pontrjagin classes see [171]) , the *L-class*

$$\mathcal{L}(M) = \mathcal{L}(p_1(M), p_2(M), \ldots) \in \bigoplus_{i\geq 0} H^{4i}(M;\mathbb{Q}).$$

Its i-th component is denoted by

$$\mathcal{L}_i(M) = \mathcal{L}_i(p_1(M), p_2(M), \ldots, p_i(M)) \in H^{4i}(M;\mathbb{Q}).$$

The famous *Signature Theorem* of Hirzebruch says that the evaluation of $\mathcal{L}_k(M)$ on the fundamental class [M] gives the signature of a $4k$-dimensional manifold M:

$$\mathrm{sign}(M) = \langle \mathcal{L}_k(p_1(M), \ldots, p_k(M)), [M]\rangle.$$

One can show that a polynomial in the Pontrjagin classes gives a homotopy invariant if and only if it is a multiple of the k-th L-class.

This sheds light on the homotopy properties of the polynomial $\mathcal{L}_k(M)$ of a $4k$-dimensional manifold M. But what can one say about the other polynomials $\mathcal{L}_1(M), \mathcal{L}_2(M), \mathcal{L}_3(M), \ldots$? Understanding $\mathcal{L}_i(M)$ is — by Poincaré duality — equivalent to understanding the numerical invariants

$$\langle x \cup \mathcal{L}_i(M), [M]\rangle \in \mathbb{Q} \tag{0.1}$$

for all $x \in H^{n-4i}(M)$, where $n = \dim(M)$. One may ask whether these numerical invariants are homotopy invariant in the following sense: If $g\colon N \to M$ is an orientation preserving homotopy equivalence, then

$$\langle x \cup \mathcal{L}_i(M), [M]\rangle = \langle g^*(x) \cup \mathcal{L}_i(N), [N]\rangle. \tag{0.2}$$

In general, these numerical invariants are not homotopy invariants. The Signature Theorem implies that the expression (0.1) is homotopy invariant for all $x \in H^0(M;\mathbb{Q})$. Novikov proved the remarkable result in the 1960s that for $\dim(M) = 4k+1$ and $x \in H^1(M)$ the expression (0.1) is homotopy invariant. This motivated Novikov to state the following conjecture.

Let G be a group. Denote by BG its *classifying space* which is up to homotopy uniquely determined by the property that it is an aspherical CW-complex with G as fundamental group. Novikov conjectured that the numerical expression

$$\langle f^*(x) \cup \mathcal{L}_i(M), [M] \rangle \in \mathbb{Q} \tag{0.3}$$

is *homotopy invariant* for every map $f\colon M \to BG$ from a closed oriented n-dimensional manifold M to BG and every class $x \in H^{n-4i}(M;\mathbb{Q})$. More precisely, the famous *Novikov Conjecture* says that if $f'\colon M' \to K$ is another map and $g\colon M \to M'$ is an orientation preserving homotopy equivalence such that $f' \circ g$ is homotopic to f, then

$$\langle f^*(x) \cup \mathcal{L}_i(M), [M] \rangle = \langle (f')^*(x) \cup \mathcal{L}_i(M'), [M']) \rangle .$$

Notice that Novikov's result that (0.2) holds in the case $\dim(M) = 4k+1$ and $x \in H^1(M)$ is a special case of the Novikov Conjecture above since S^1 is a model for $B\mathbb{Z}$ and a cohomology class $x \in H^1(M)$ is the same as a homotopy class of maps $f\colon M \to S^1$, the correspondence is given by associating to the homotopy class of $f\colon M \to S^1$ the pullback $f^*(x)$, where x is a generator of $H^1(S^1)$.

Looking at this conjecture in a naive way one does not see a philosophical reason why it should be true. Even in the case of the polynomial $\mathcal{L}_k$, where $4k$ is the dimension of a manifold, the proof cannot be understood without the signature theorem translating the L-class to a cohomological invariant, the signature. In this situation it is natural to ask for other homotopy invariants (instead of the signature) hoping that one can interpret the expressions (0.3) occurring in the Novikov Conjecture in terms of these invariants. These expressions (0.3) are called *higher signatures*. One can actually express them as signature of certain submanifolds. But this point of view does not give homotopy invariants.

It is natural to collect all higher signatures and form from them a single invariant. This can be done, namely, one considers

$$\operatorname{sign}^G(M,f) := f_*(\mathcal{L}(M) \cap [M]) \in \bigoplus_{i \in \mathbb{Z}, i \geq 0} H_{m-4i}(BG;\mathbb{Q}),$$

the image of the Poincaré dual of the L-class under the map induced from f. An approach to proving the Novikov Conjecture could be to construct a homomorphism

$$A^G\colon \bigoplus_{i \in \mathbb{Z}, i \geq 0} H_{m-4i}(BG;\mathbb{Q}) \to L(G)$$

where $L(G)$ is some abelian group, such that $A^G(\operatorname{sign}_G(M))$ is a homotopy invariant. Then the Novikov Conjecture would follow if the map A^G is injective. Such maps will be given by so-called *assembly maps*.

The construction of such a map is rather complicated. A large part of these lecture notes treats the background needed to construct such a map. In particular, one needs the full machinery of surgery theory. We will give an introduction to this important theory. Roughly speaking, surgery deals with the following problem. Let W be a compact m-dimensional manifold whose boundary is either empty or consists of two components M_0 and M_1 and $f\colon W \to X$ a map to a finite CW-complex. If the boundary of W is not empty, we assume that f restricted to M_0 and M_1 is a homotopy equivalence. Then X is a so-called *Poincaré complex*, something we also require if the boundary of W is empty. The question is whether we can replace W and f by W' and f' (bordant to (W, f)) such that f' is a homotopy equivalence. If the boundary of W is not empty, then W' is an *h-cobordism* between M_0 and M_1. In general it is not possible to replace (W, f) by (W', f') with f' a homotopy equivalence. Wall has defined abelian groups $L^h_m(\pi_1(X))$ and an obstruction $\theta(W, f) \in L^h_m(\pi_1(X))$ whose vanishing is a necessary and sufficient condition for replacing (W, f) by (W', f') with f' a homotopy equivalence, if $m > 4$. One actually needs some more control, namely a so-called normal structure on W. All this is explained in Chapters 2, 10–14 and Chapter 17.

Why is it so interesting to obtain an h-cobordism? If X is simply-connected, and the dimension of W is greater than five, the celebrated *h-cobordism theorem* of Smale says that an h-cobordism W is diffeomorphic to the cylinder over M_0. In particular, M_0 and M_1 are diffeomorphic. There is a corresponding result for topological manifolds. In the situation which is relevant for the Novikov Conjecture, X is not simply-connected and then the h-cobordism theorem does not hold. There is an obstruction, the *Whitehead torsion*, sitting in the *Whitehead group* which is closely related to the algebraic K_1-group. If the dimension of the h-cobordism W is larger than five, then the vanishing of this obstruction is necessary and sufficient for W to be diffeomorphic to the cylinder. This is called the *s-cobordism theorem*. The Whitehead group, the obstruction and the idea of the proof of the s-cobordism theorem are treated in Chapters 5–8.

In Chapters 15–16 we define the assembly map and apply it to prove the Novikov Conjecture for finitely-generated free abelian groups.

What we have presented so far summarizes and explains information which was known around 1970. To get a feeling for how useful the Novikov Conjecture is, we apply it to some classification problems in low dimensions (see Chapter 0).

In the rest of the lecture notes we present some of the most important concepts and results concerning the Novikov Conjecture and other closely related conjectures dating from after 1970. This starts with an introduction to spectra (see Chapter 18) and continues with classifying spaces of families, a generalization of aspherical spaces (see Chapter 19). With this we have prepared a frame in which not only the Novikov Conjecture but other similar and very important conjectures can be formulated: the *Farrell–Jones* and the *Baum–Connes Conjectures*. After introducing equivariant homology theories in Chapter 20, these conjectures and their relation to the Novikov Conjecture are discussed in Chapters 21–23. Finally, these lecture notes are finished by Chapter 24 called "Miscellaneous" in which the status of the conjectures is summarized and methods and proofs are presented.

It is interesting to speculate whether the Novikov Conjecture holds for all groups. No counterexamples are known to the authors. An interesting article expressing doubts was published by Gromov [102].

We have added a collection of exercises and hints for their solutions.

From the amount of material presented in these lecture notes it is obvious, that we cannot present all of the details. We have tried to explain those things which are realistic for the very young participants of the seminar to master and we have only said a few words (if anything at all) at other places. People who want to understand the details of this fascinating theory will have to consult other books and often the original literature. We hope that they will find our lecture notes useful, since we explain some of the central ideas and give a guide for learning the beautiful mathematics related to the Novikov Conjecture and other closely related conjectures and results.

We would like to thank the participants of this seminar for their interest and many stimulating discussions and Mathematisches Forschungsinstitut Oberwolfach for providing excellent conditions for such a seminar. We also would like to thank Andrew Ranicki for carefully reading a draft of this notes and many useful comments.

Contents

Chapter 0

A Motivating Problem

The classification of manifolds is one of the central problems in mathematics. Since a complete answer is (at least for manifolds of dimension ≥ 4) not possible, one firstly has to fix certain invariants in such a way that the classification is in principle possible. The reason why the classification of manifolds is impossible is reduced to the impossibility of classifying their fundamental groups. Thus as a first invariant one has to fix the fundamental group. Then the optimal answer would be to find invariants which determine the diffeomorphism (homeomorphism or homotopy) type. In recent years, low dimensional manifolds (in dimension up to 7) occurred in various mathematical and non-mathematical contexts. We motivate the Novikov Conjecture by considering the following problem:

Problem 0.1 (Classification of manifolds in low dimensions with $\pi_1(M) \cong \mathbb{Z}^2$ and $\pi_2(M) = 0$). *Classify all connected closed orientable manifolds M in dimensions ≤ 6 with fundamental group $\pi_1(M) \cong \mathbb{Z}\oplus\mathbb{Z}$ and second homotopy group $\pi_2(M) = 0$ up to*

(1) *homotopy equivalence;*

(2) *homeomorphism;*

(3) *diffeomorphism.*

Here and in the following we always mean *orientation preserving maps.*

0.1 Dimensions ≤ 4

Since all closed connected 1-manifolds are diffeomorphic to S^1, there is no example in dimension 1.

In dimension 2 there is only one such manifold, the torus $T^2 = S^1 \times S^1$. Here the classification up to the relations i)–iii) agree.

In dimension 3 there is no manifold with fundamental group $\mathbb{Z}\oplus\mathbb{Z}$. The reason is that the classifying map of the universal covering of such a 3-manifold $f\colon M \to T^2$ is 3-connected. Hence it induces isomorphisms $H_p(f)\colon H_p(M) \xrightarrow{\cong} H_p(T^2)$ and $H^p(f)\colon H^p(T^2) \xrightarrow{\cong} H^p(M)$ for $p \leq 2$. Poincaré duality implies $H^1(M) \cong H_2(M)$. This yields a contradiction since $H_2(T^2) = \mathbb{Z}$ and $H_1(T^2) = \mathbb{Z}^2$.

There is also no such manifold M in dimension 4 by the following argument. As above for 3-manifolds we conclude that $H_2(M) \cong \mathbb{Z}$. Poincaré duality implies for the Euler characteristic $\chi(M) = 1$. Now we note that all finite coverings of M are again manifolds of the type we investigate. Namely the fundamental group is a subgroup of finite index in $\mathbb{Z} \oplus \mathbb{Z}$ and so isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$. And the higher homotopy groups of a covering do not change. Now consider a subgroup of index $k > 1$ in $\pi_1(M)$ and let N be the corresponding covering. Then $\chi(N) = k \cdot \chi(M)$. Since N is a manifold under consideration we have $\chi(N) = 1$. This leads to a contradiction.

0.2 Dimension 6

In dimension 6 one has an obvious example, namely $T^2 \times S^4$. But there are many more examples coming from the following construction.

Example 0.2 (Constructing manifolds by surgery). We start with a simply connected smooth 4-manifold M with trivial second Stiefel–Whitney class $w_2(M)$ and consider $T^2 \times M$. Then we choose disjoint embeddings $(S^2 \times D^4)_i$ into $T^2 \times M$ representing a basis of $\pi_2(M) \cong \pi_2(T^2 \times M)$. For this we first choose maps from S^2 to $T^2 \times M$ representing a basis. The Whitney Embedding Theorem implies that we can choose these maps as disjoint smooth embeddings. Finally we note that since $w_2(T^2 \times M) = 0$, the normal bundle of these embeddings is trivial and we use a tubular neighbourhood to construct the desired embeddings. Now we form a new manifold by deleting the interiors of these embeddings and glueing in $D^3 \times S^3$ to each deleted component. We denote the resulting manifold by $N(M)$. This cutting and pasting process is called *surgery*. Using standard considerations in algebraic topology one shows that $N(M)$ is an oriented manifold with $\pi_1 \cong \mathbb{Z} \oplus \mathbb{Z}$, $\pi_2 = 0$ and $w_2 = 0$ (see Exercise 0.1). We will study surgery in later chapters systematically.

Example 0.3 (A higher signature). We introduce the following invariant for the 6-manifolds N under consideration. The second cohomology is isomorphic to $\mathbb{Z}$, and we choose a generator $x \in H^2(N)$. This generator is well defined up to sign. Let $[M]$ be the fundamental class in $H_6(M)$. Taking the cup product with the Pontrjagin class and evaluating on $[M]$ gives our invariant:

$$\pm\langle x \cup p_1(N), [N]\rangle \ \in \mathbb{Z} \tag{0.4}$$

which is unique up to a sign $\pm$. It is easy to see (see Exercise 0.2) that for the manifold $N(M)$ constructed above this invariant agrees with the first Pontrjagin class of M evaluated at $[M]$ up to sign:

$$\pm \langle x \cup p_1(N(M)), [N(M)]\rangle \quad = \quad \pm\langle p_1(M), [M]\rangle. \tag{0.5}$$

The values of $\langle p_1(M), [M]\rangle$ for the different simply connected smooth 4-manifolds are known: Every integer divisible by 48 occurs [231].

We want to understand the relevance of this invariant. We firstly note that it is unchanged if we take the connected sum with $S^3 \times S^3$. Thus it is an invariant of the stable diffeomorphism type, where we call two closed manifolds M and N of dimension $2k$ *stably diffeomorphic*, if there exist integers p and q, such that $M\sharp p(S^k \times S^k)$ is diffeomorphic to $N\sharp q(S^k \times S^k)$, i.e., the manifolds M and N are diffeomorphic after taking the connected sum with p resp. q copies of $S^k \times S^k$. The relevance of the invariant $\langle x \cup p_1(N), [N]\rangle$ is demonstrated by the following result

Theorem 0.6 (Stable Classification of Certain Six-Dimensional Manifolds). *Two smooth 6-dimensional closed orientable manifolds M and N with $\pi_1(M)\cong\pi_1(N)\cong \mathbb{Z}\oplus\mathbb{Z}$ and $\pi_2(M) = \pi_2(N) = 0$ are stably diffeomorphic if and only if*

(1) *in both cases w_2 vanishes or does not vanish;*

(2) $\pm\langle x \cup p_1(M), [M]\rangle \; = \; \pm \langle x \cup p_1(N), [N]\rangle$.

We will give the proof of this result in Chapter 14. In our context this result leads to the following obvious questions: Is the second invariant also a stable homeomorphism or stable homotopy invariant? Here we define *stably homeomorphic* and *stably homotopy equivalent* in analogy to the definition of stably diffeomorphic by replacing in this definition diffeomorphic by homeomorphic or homotopy equivalent.

The answer is in both cases non-trivial. For homeomorphisms we pass from the Pontrjagin class $p_1(M) \in H^4(M)$ to the rational Pontrjagin class $p_1(M;\mathbb{Q}) \in H^4(N;\mathbb{Q})$. Since $H^4(N)$ is torsionfree we do not lose any information. Then we apply a deep result by Novikov (see Theorem 1.5) saying that the rational Pontrjagin classes are homeomorphism invariants and so stable homeomorphism invariants.

The rational Pontrjagin classes are in general not homotopy invariants (see Example 1.6). But Novikov conjectured that certain numerical invariants, the so-called higher signatures, built from the rational Pontrjagin classes and cohomology classes of the fundamental group are homotopy invariants. The invariant occurring in Theorem 0.6 is one of these invariants. (We will give a proof for free abelian groups in Chapter 16).

It should be noted that in contrast to the Pontrjagin classes the Stiefel–Whitney classes of a manifold are homotopy invariants. Thus the condition $w_2 = 0$ or $w_2 \neq 0$ is invariant under (stable) homotopy equivalences. Thus we conclude

Corollary 0.7. *For two smooth 6-dimensional closed orientable manifolds M and N with $\pi_1(M) \cong \pi_1(N) \cong \mathbb{Z} \oplus \mathbb{Z}$ and $\pi_2(M) = \pi_2(N) = 0$ the classifications up to stable diffeomorphism, stable homeomorphism and stable homotopy equivalence agree. In other words, the invariants from Theorem 0.6 determine also the stable homeomorphism and stable homotopy type.*

Remark 0.8 (Role of the Novikov Conjecture). The following formulation explains the surprising role of the Novikov Conjecture. In general the homotopy classification is a simpler question than the homeomorphism or diffeomorphism classification which one can attack by methods of classical homotopy theory. For the 6-manifolds under consideration the Novikov Conjecture implies that the stable homotopy type determines the stable homeomorphism and even stable diffeomorphism type of these smooth manifolds.

0.3 Dimension 5

Now we study the manifolds in dimension 5. In dimension 5 there are at least two such manifolds, namely $T^2 \times S^3$ and the sphere bundle of the non-trivial oriented 4-dimensional vector bundle over T^2. These manifolds are not homotopy equivalent (see Exercise 0.3). Moreover, using standard techniques from homotopy theory one can show that there are precisely two homotopy types of manifolds under consideration, which are given by these two bundles. The next obvious question is the determination of the homeomorphism and diffeomorphism type of these manifolds. One can show that the diffeomorphism type is determined by the first Pontrjagin class, and since this is a homeomorphism invariant (by Novikov's result mentioned above), this also determines the homeomorphism type. But which values can the Pontrjagin class take? Here again the Novikov Conjecture comes into play. It implies in our situation that the first Pontrjagin class is a homotopy invariant. Since we know all homotopy types and in the examples above the Pontrjagin class is trivial, we conclude that the Pontrjagin class is zero for our manifolds. Thus we have again a surprising result: The homotopy type of these 5-manifolds determines the homeomorphism (and actually diffeomorphism) type! For detailed arguments and more results we refer to [136].

Remark 0.9 (Other fundamental groups). The Novikov Conjecture is also valid for all fundamental groups G of closed oriented surfaces. The proof of Theorem 0.6 also holds for these fundamental groups so that Corollary 0.7 can also be generalized to these fundamental groups. We will investigate these 6-manifolds further in [136].