SPRINGER BRIEFS IN APPLIED SCIENCES AND TECHNOLOGY · COMPUTATIONAL MECHANICS

Severino P. C. Marques · Guillermo J. Creus

Computational Viscoelasticity

Springer

Severino P. C. Marques · Guillermo J. Creus

Computational Viscoelasticity

Springer

Severino P. C. Marques
Centro de Tecnologia
Universidade Federal de Alagoas
Maceió, Alagoas
Brazil
e-mail: smarques@lccv.ufal.br

Guillermo J. Creus
ILEA
Universidade Federal do Rio Grande do Sul
Porto Alegre
Rio Grande do Sul
Brazil
e-mail: creus@ufrgs.br

ISSN 2191-5342
e-ISSN 2191-5350
ISBN 978-3-642-25310-2
e-ISBN 978-3-642-25311-9
DOI 10.1007/978-3-642-25311-9
Springer Heidelberg Dordrecht London New York

Library of Congress Control Number: 2011943323

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

To my wife, Dilze, and my children,
Fernando, Gustavo and Clarissa

S. P. C. Marques

To my wife, Susana, and my children,
Guillermo, Tomás, Amalia and Ana

G. J. Creus

Preface

This book develops a presentation of viscoelasticity theory oriented toward numerical applications. It is our hope that it will be useful both as a textbook for graduate courses and as a reference volume for engineers and researchers.The book is structured in twelve chapters. The first eight chapters introduce basic concepts and theoretical ideas about the viscoelastic response of solids. They cover constitutive relations in integral and differential form, influence of temperature, age and finite strain. These topics were selected aiming to make the access to the computational viscoelastic formulations easier. It is assumed that the reader has a background in mathematics and mechanics at the undergraduate level. In the last five chapters a more advanced experience may be needed.

The remaining chapters address the numerical formulation of viscoelastic problems using finite element, boundary element and finite volume methods. Chapter 9 presents viscoelastic finite element procedures formulated on a total Lagrangian description for large displacements and rotations with small strains. Two alternative boundary element procedures for the solution of problems in linear viscoelasticity are reviewed in Chap. 10: the solution in the Laplace transformed domain and the use of a general inelastic formulation. Chapter 11 presents a two-dimensional approach for linear viscoelastic solids using a finite volume framework. Together with the theoretical formulations, worked examples are presented throughout the text. Finally, in Chap. 12, further examples, to be solved with the software Abaqus, are proposed and developed. The book concludes with three Appendices which contain auxiliary expressions in mathematics and mechanics.

Several colleagues and students provided essential help. We mention here professors L. A. B. Cunda (FURG), B. F. Oliveira (UFRGS) and Paul Partridge (UnB). D. La Porta, D. Palmer and R. Sprunger (SIMULIA) helped with the Abaqus examples, Litha Bacci draw the figures and Joice de Brito e Cunha checked the English text. This work is the result of collaboration between the Federal University of Alagoas (UFAL) and the Federal University of Rio Grande do Sul (UFRGS) with the financial support of the Brazilian Agency CAPES through PROCAD program. The continuous support of our research by the Brazilian Agency CNPq is also gratefully acknowledged.

Federal University of Alagoas-Brazil — Severino P. C. Marques
Federal University of Rio Grande do Sul-Brazil — Guillermo J. Creus

Contents

Chapter 1
Introduction

1.1 Historical Context

First studies. It took time to discover that the properties of important materials lay outside the classical limits of Hookean elastic solids and Newtonian viscous fluids. Tests on the mechanical properties of silk threads, performed in 1835 by Wilhelm Weber, showed that solid behavior could have viscous components. Later, in 1867, James Clerk Maxwell introduced elastic properties in the description of fluids. Boltzmann developed in 1874 the formulation for linear viscoelasticity. Using the superposition of effects, he showed that the strain at time t in response to a general time-dependent stress history $\sigma(t)$ can be written as the sum (or integral) of terms that involve the strain response to a step loading. The mathematician Vito Volterra [7] developed the theory of functional and integral equations adequate to model viscoelastic phenomena. Differential and integral representations of viscoelasticity [2, 3, 6] are addressed in Chap. 2.

Further developments and problems. The developments in the first half of the twentieth century were slow and important advances in theoretical and experimental rheology took place only after World War II. New materials, such as polymers and composite materials [1, 5] posed new problems, particularly the need to solve boundary value problems in varying conditions of temperature and humidity. Chapter 3 introduces the state variables formalism, important for efficient computation and Chap. 4 extends the viscoelastic formulation to three dimensional situations. The effect of temperature is studied in Chap. 6, and the Laplace transform technique, used to solve boundary value problems, is reviewed in Chap. 5. In the analysis of materials such as rubber, soft polymers and biological tissues strains are large and it is necessary to dispense with the infinitesimal strain theory. To maintain objectivity in the presence of large rotations, measures like the Cauchy-Green tensor for strain and the Piola–Kirchhoff tensors for stress are introduced. This formulation is reviewed in Chap. 8. Biological tissues [4],

S. P. C. Marques and G. J. Creus, *Computational Viscoelasticity*,
SpringerBriefs in Computational Mechanics, DOI: 10.1007/978-3-642-25311-9_1,

polymers and other important materials show a mechanical behavior that depends on age. This subject is introduced in Chap. 7.
Computational viscoelasticity. Digital computers revolutionized the practice of many areas of engineering and science, and solid mechanics was among the first fields to use them. Many computational techniques have been used in this field, but the one that emerged in the 1970s as the most widely adopted is the Finite Element Method. This method was developed and put to practical use for the analysis of aeronautical structures by Ray W. Clough and J. H. Argyris. In the most common version of the Finite-Element Method, the domain to be analyzed is divided into elements, and the displacement field within each element is interpolated in terms of the displacements at the nodes. From the displacements, strains and stresses are calculated in terms of nodal displacements. The equilibrium equations expressed through the principle of virtual work generate a system of simultaneous equations to be solved by the computer. With the Finite Elements Method, for the first time, real problems could be analyzed considering the actual geometry and material properties. First bar structures and small strain elasticity and then geometrical and physical nonlinear problems were addressed and solved. Lately, both the Boundary Element Method, that reduces the dimension of the problems and provides very precise results, and the Finite Volume Method, which seems to be very efficient for the study of non-homogeneous solids, were developed. These numerical procedures are analyzed in the second part of this book, Chaps. 9, 10, 11. In Chap. 12 some computational examples and exercises are included, using Abaqus software.

1.2 Basic Experimental Results

The characteristic feature of viscoelastic behavior is the essential role played by time. Viscoelastic materials under constant stress increase their deformation with time, while, under constant strain, show stresses that decrease with time. Figure 1.1 indicates the behavior of a typical viscoelastic material in a *creep test* characterized by the application of a constant stress σ_0 at a time τ_0.
Using the unit step function $H(t)$, defined in Appendix A, we may write this stress history as

$$\sigma(t) = \sigma_0 H(t - \tau_0) \tag{1.1}$$

which defines both the value of the applied stress and the time of its application. In a creep test we measure an elastic strain component ε^e(instantaneous) and a creep (delayed) component ε^c. The latter is the one that increases with time and characterizes viscoelastic behavior. The deformation that remains after $\tau > \tau_0$ characterizes hysteresis.

Removing the applied stress at time $\tau_1 > \tau_0$, that is, considering the stress history

$$\sigma(t) = \sigma_0 H(t - \tau_0) - \sigma_0 H(t - \tau_1) \tag{1.2}$$

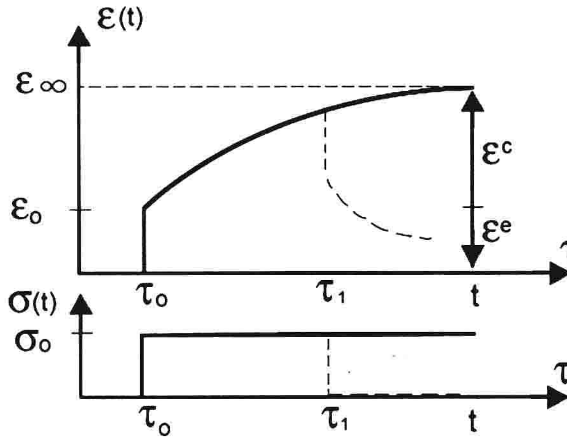

Fig. 1.1 Creep test of a viscoelastic solid: histories of stress and strain. Full line, loading; dotted line, unloading

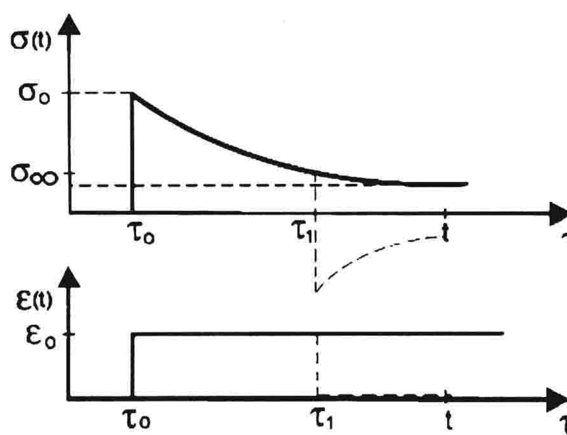

Fig. 1.2 Relaxation test of a viscoelastic solid: histories of stress and strain. Full line: loading; dotted line: unloading

we obtain for $t > \tau_1$ the strain history shown by the dotted line in Fig. 1.1. The deformation reduction upon unloading is known as *creep recovery*.

In a *relaxation test* we have the material subjected to an imposed constant deformation

$$\varepsilon(t) = \varepsilon_0 H(t - \tau_0) \tag{1.3}$$

and we measure the stress $\sigma(t)$ that is needed to keep strain at the constant value ε_0. We observe that $\sigma(t)$ diminishes progressively, as indicated by the stress history in Fig. 1.2. Removing the applied deformation at time τ_1, that is, considering the history

$$\varepsilon(t) = \varepsilon_0 H(t - \tau_0) - \varepsilon_0 H(t - \tau_1) \tag{1.4}$$

we obtain the stress history shown in dotted lines. It is interesting to observe that in this case we may have a change in the sign of the resultant stress; this fact may be of importance for materials with different strengths in tension and compression.

1.3 Constitutive Relations

The general principles of mechanics (e.g. equilibrium and compatibility equations, thermodynamic principles) are valid for all materials. The characteristic properties of each material are specified by its constitutive equations.

A *constitutive equation* is a relation between forces and deformations. In popular terms, the forces applied to a body "cause" it to deform and the quality and amount of deformation varies according to the nature of the body. In the present context (small deformation analysis) stresses and deformations are conveniently represented by *Cauchy stress* σ and *infinitesimal strain* ε. Constitutive relations will be firstly discussed in a uniaxial setting. The extension to the multiaxial case will be analyzed in Chap. 4, and the extension to finite strains, in Chap. 8. A more precise definition of the concepts of strain and stress can be found in Appendix B and references there.

In practice, constitutive relations are firstly suggested by experiments and then established by means of mathematical equations. New experiments, new materials, new applications, lead to new more refined or more sophisticated models.

1.3.1 Dependence on Time History; Elastic and Viscoelastic Materials

During a typical experience, we apply to a specimen a stress history $\sigma(t)$, variable in time ($\tau_0 \le t \le \infty$) and we measure the corresponding strain history $\varepsilon(t)$. We may also apply a deformation history $\varepsilon(t)$ and measure the resulting stresses $\sigma(t)$, because the choice of the controlled variable is a matter of experimental convenience. For an arbitrary stress history, the strain at time t will depend, in general, upon all the values of stress in the time interval of the experiment, so that we can write

$$\varepsilon(t) = \mathfrak{D}\underset{\tau=\tau_0}{\overset{\tau=t}{\{\sigma(\tau)\}}} \tag{1.5}$$

where $\mathfrak{D}$ indicates a functional $\mathfrak{D} : C(\tau_0, t) \Rightarrow R$ while $C(\tau_o, t)$ and R indicate respectively the set of continuous functions defined in the interval $[\tau_0, t]$ and the set of real numbers. Eq. (1.5) indicates that the value of ε at time t depends on all the values of $\sigma(\tau)$ for τ varying between τ_0 and t. τ_0 is an arbitrary initial time, so that $\sigma(t) = 0$ and $\varepsilon(t) = 0$ for $t < \tau_0$.

Similarly, we can write

$$\sigma(t) = \mathcal{E}\{\varepsilon(\tau)\}_{\tau=\tau_0}^{\tau=t} \tag{1.6}$$

Notice here the existence of two symbols representing time. t is used to represent the time of interest. For example, in (1.5) we are interested in the deformation at time t. This deformation depends on all the stresses applied to the material in different instants up to time t. To avoid confusion, we use another symbol, τ, to represent those instants. τ is a dummy variable that runs in the interval that ends in t.

A different functional $\mathfrak{D}$ corresponds to each class of material. For example, in elastic materials the deformation at time t depends on the value of the stress at the same time t: instantaneous, non hereditary response. Elastic materials have a very short memory: they recall only the present stress, when $\tau = t$. In this case, the functional in (1.5) is reduced to an ordinary function and

$$\varepsilon(t) = D(\sigma(t)). \tag{1.7}$$

If the material is linearly elastic (1.7) may be still simplified to

$$\varepsilon(t) = D\sigma(t) \tag{1.8}$$

where D is now a constant factor, the *elastic compliance*, which is the inverse of the *elastic modulus E*.

On the other hand, viscoelastic materials are characterized by a dependence on the whole history of the deformation process, and their constitutive relations must have the functional structure indicated in (1.5) and (1.6). Considering for example the creep test, as described in Sect. 1.2, we see that its result may be expressed in the form (1.5). In this particular case, the argument of the functional is completely determined once we know the values of σ_0, t and τ, being τ a generic time for loading. Thus, creep tests may be characterized by a functional whose argument is formed by step functions, or, equivalently, by a function of three variables

$$\varepsilon(t) = D(\sigma_0, t, \tau_0) \tag{1.9}$$

We have already seen how this function depends on t; now we will analyze its dependence on σ_0(the stress applied in the creep test) and τ_0(the time at which the creep test begins).

1.3.2 Dependence on Stress: Linearity

Figure 1.3a indicates the stress and strain histories for creep tests of a typical material at different stress levels. We see that for small stresses the deformations tend to stabilize, while for high stresses they grow at an increasing rate. This type of behavior is usual in concrete, polymers and many other viscoelastic materials. Figure 1.3b shows isochronous curves, that are obtained from Fig. 1.3a by setting

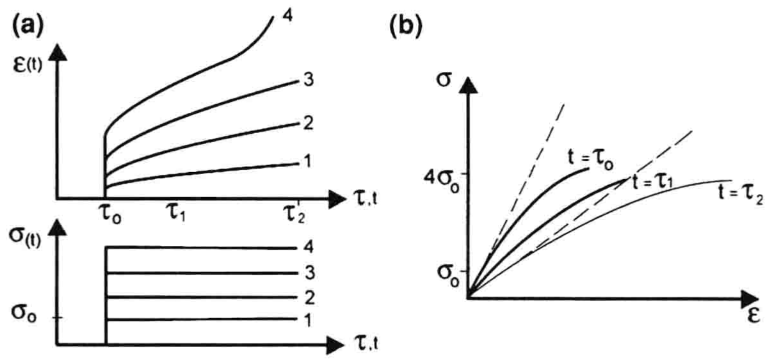

Fig. 1.3 **a** Creep tests with different values of stress and the corresponding strain histories. **b** isochronous curves corresponding to the tests in (a)

τ as a parameter. We can do this graphically just by choosing values of τ in Fig. 1.3a and determining the corresponding values of σ and ε. These *isochronous* (from iso: equal, chronos: time) curves are pseudo stress–strain relations, but of course are valid only in reference to creep tests.

In the case of Fig. 1.3, the threshold of nonlinearity is about $2\sigma_0$. Its precise location depends on the accepted tolerance. Linearity in this context may be characterized by superposition. Consider arbitrary stress histories of the type

$$\sigma(\tau) = \sigma_1(\tau) + \sigma_2(\tau)\,;\ \tau \in [\tau_0, t] \tag{1.10}$$

If they give rise to strain histories that can be expressed as

$$\varepsilon(\tau) = \varepsilon_1(\tau) + \varepsilon_2(\tau) \tag{1.11}$$

where $\varepsilon_1(\tau)$ and $\varepsilon_2(\tau)$ are the strain histories corresponding to $\sigma_1(\tau)$ and $\sigma_2(\tau)$ separately, we say that the material is linear. Linear behavior is also referred to as obeying the "*Principle of superposition in viscoelasticity*" or "*Boltzmann principle*". To check linear behavior experimentally, step functions are usually used. The representation of nonlinear viscoelasticity is addressed in Chap. 8.

On the linear range, we may write (1.9) in the form

$$\varepsilon(t) = \sigma_0 D(t, \tau) \tag{1.12}$$

where $D(t, \tau)$, *the specific creep function* or *creep compliance*, defined as the response at time t to a unit step of stress applied at time τ, fully characterizes the behavior of a linear viscoelastic material.

In material testing it is usual to use uniaxial tension or compression loading applying a strain history with constant rate $\varepsilon(t) = vt$. An elastic material will show a stress history also with constant rate. This is not the case when the material is *linear* viscoelastic. A typical result is shown in Fig. 1.4. Stress-time and stress–strain relations are not linear except for very slow ($v \to 0$) or very fast ($v \to \infty$) loading rates.

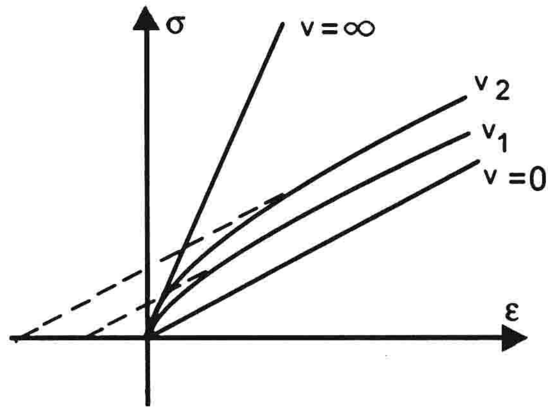

Fig. 1.4 Loading of a linear viscoelastic material (standard model) with constant strain rate

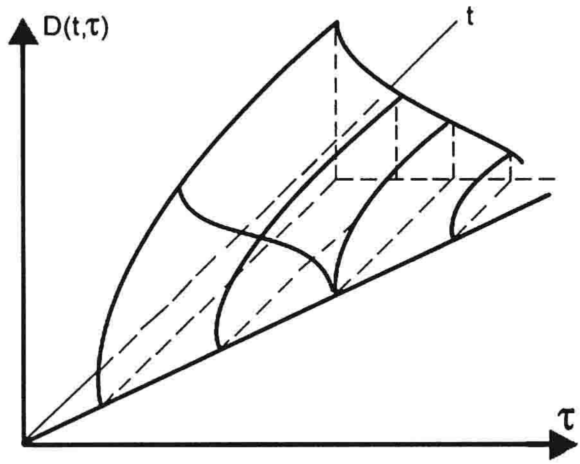

Fig. 1.5 Surface representing the creep function $D(t, \tau)$ for a material that hardens with age (i.e. concrete)

When the limiting value of $D(t, \tau)$ is finite, i.e., $\lim_{t \to \infty} D(t, \tau) = M(\tau) < \infty$ we say that the material is asymptotically stable. Sometimes, asymptotically stable materials are referred to as solids, while those materials for which $D(t, \tau)$ grows indefinitely are called fluids. For stable materials we have

$$\frac{\partial D(t, \tau)}{\partial (t - \tau)} < 0 \tag{1.13}$$

1.3.3 Dependence on Age: Aging

We call *aging* the change in the mechanical properties of a given material due to its age, where age is the time period between some origin more or less arbitrarily

established and the time of observation. Concrete is a typical example of an aging material. From the moment of casting (taken usually as age zero) it begins to increase its strength and to decrease its deformability. The function $D(t, \tau)$ that indicates the specific creep has for concrete the form indicated in Fig. 1.5. Notice that $D(t, \tau) = 0$ for $t < \tau$.
Frequently, the concept of aging involves other influences in addition to elapsed time. Aging is different according to the environmental conditions in which the material ages. In the case of concrete, humidity and temperature are important. In the case of polymers factors such as temperature, humidity, UV radiation, etc., make a difference. In the case of a viscoelastic material without aging we have $D(t + a, \tau + a) = D(t, \tau), \forall a$. Thus, for $a = -\tau$, we can write

$$D(t,\tau) = D(t - \tau) \tag{1.14}$$

Non-aging materials represent a special (very important) case of viscoelastic materials. Additional formulations and examples for the *aging* case are given in Chap. 7.

1.4 State Variables Formulation

Besides the functional representation described in Sect. 1.3, a state variables representation may be used with some advantages in viscoelasticity as well as in plasticity and damage mechanics. An advantage of the *state variable approach* is that physical theories, and micro-structural information, may be introduced directly in the formulation of the evolution equations. Another one is that it leads to more efficient numerical procedures. This formulation will be introduced in Chap. 3.

1.5 Computational Viscoelasticity

Because of mathematical difficulties few real problems in viscoelasticity have analytical solution. As in many other areas of science, the use of numerical analyses and digital computers had a great impact in this field. Procedures based on techniques as Finite Elements, and more recently, Boundary Elements and Finite Volumes allow the analysis of complex bodies and structures made of linear and nonlinear viscoelastic materials. In Chaps. 9, 10, 11 of this book, these numerical procedures are described. To allow the reader to have some practice with computational procedures a few examples using the well known commercial software Abaqus are given.