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Volume 2

Edited by
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and Alex Wilkie

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Model Theory with Applications to Algebra and Analysis

Volume 2

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Preface

These two volumes contain both expository and research papers in the general area of model theory and its applications to algebra and analysis. The volumes grew out of the semester on "Model Theory and Applications to Algebra and Analysis" which took place at the Isaac Newton Institute (INI), Cambridge, from January to July 2005. We, the editors, were also the organizers of the programme. The contributors have been selected from among the participants and their papers reflect many of the achievements and advances obtained during the programme. Also some of the expository papers are based on tutorials given at the March-April 2005 training workshop. We take this opportunity, both as editors of these volumes and organizers of the MAA programme, to thank the Isaac Newton Institute and its staff for supporting our programme and providing a perfect environment for mathematical research and collaboration.

The INI semester saw activity and progress in essentially all areas on the "applied" side of model theory: o-minimality, motivic integration, groups of finite Morley rank, and connections with number theory and geometry. With the exception of motivic integration and valued fields, these topics are well represented in the two volumes.

The collection of papers is more or less divided into (overlapping) themes, together with a few singularities. Aspects of the interaction between stability theory, differential and difference equations, and number theory, appear in the first six papers of volume I. The first paper, based on Pillay's workshop tutorial, can also serve as a fast introduction to model theory for the general reader, although it quickly moves to an account of Mordell-Lang for function fields in characteristic 0. The "arithmetic of differential equations" figures strongly in Pillay's paper on the Grothendieck-Katz conjecture and its nonlinear generalizations, as well as in Bertrand's paper which initiates the investigation of versions of Ax-Schanuel for nonisoconstant semiabelian varieties over function fields. The Galois theory of difference equations is rather a hot topic and the Chatzidakis-Hardouin-Singer paper compares definitions and concepts that have arisen in algebra, analysis, and model theory.

Interactions of complex analytic geometry with model theory and logic (in the form of stability, o-minimality, as well as decidability issues) appear in papers 7 to 10 of volume 1. The papers by Peterzil-Starchenko and Moosa-Pillay (on nonstandard complex analysis and compact Kähler manifolds respectively) are comprehensive accounts of important projects, which contain new results and set the stage for future research. In the first, o-minimality is the model-theoretic tool. In the second it is stability. Wilkie's paper characterizes the holomorphic functions locally definable from a given family of holomorphic functions, and Macintyre's paper is related to his work on the decidability of Weierstrass functions. They are both set in the o-minimal context.

The o-minimality theme is continued in papers 12 and 13 of volume 1 from a (real) geometric point of view. In particular Rolin's paper is a comprehensive account of the most modern techniques of finding o-minimal expansions of the real field.

In recent years Zilber has been exploring connections between model theory and noncommutative geometry, and in his paper in volume I he succeeds in interpreting certain "quantum algebras" as Zariski structures. Fesenko's short note contains a wealth of speculations and questions, including the use of nonstandard methods in noncommutative geometry.

Definable groups of "finite dimension" in various senses (finite Morley rank, finite SU-rank, o-minimal) figure strongly in papers 1 to 5 of volume II. Papers 1 and 2 contain new and striking general results on groups of finite Morley rank, coming out of techniques and results developed in work on the Cherlin-Zilber conjecture. Paper 3 gives an overview of a model-theoretic approach to asymptotics and measure stimulated by the analogous results and concepts for finite and pseudofinite fields. The article by Hrushovski and Wagner, on the size of the intersection of a finite subgroup of an algebraic group with a subvariety, generalizes a theorem of Pink and Larsen. Otero's paper gives a comprehensive description of work since the 1980's on groups definable in o-minimal structures. This includes an account of the positive solution to "Pillay's conjecture" on definably compact groups which was proved during the Newton semester.

Hilbert's 10th problem and its generalizations, as well as first order properties of function fields, appear in papers 6 to 8 of volume II. The Pheidas-Zahidi and Eistenträger papers are based on tutorials given at INI, and give a comprehensive account of work on Hilbert's 10th problem for the rational field and for various rings and fields of functions. Paper 8 proves among other things definability of the constant field in function fields whose constant field is "large". The three papers together give a good picture of an exciting and very active subject at the intersection of logic and number theory.

The volumes are rounded off by important papers on Hrushovski constructions, ordered abelian groups, and continuous logic. In particular the paper 10 in volume II (based again on a tutorial) is an elementary and self-contained presentation of "continuous logic" or the "model theory of metric structures" which is fast becoming an autonomous area of model theory with links to both stability and functional analysis.

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Table of contents for Volume 1

Conjugacy in groups of finite Morley rank

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Summary

We survey conjugacy results in groups of finite Morley rank, mixing unipotence, Carter, and Sylow theories in this context.

Introduction

When considering certain classes of groups one might expect conjugacy theorems, and the class of groups of finite Morley rank is not an exception to this. The study of groups of finite Morley rank is mostly motivated by the Algebricity Conjecture, formulated by G. Cherlin and B. Zilber in the late seventies, which postulates that infinite simple groups of this category are isomorphic to algebraic groups over algebraically closed fields. The model-theoretic rank involved appeared in the sixties when M. Morley proved his famous theorem on the categoricity in *any* uncountable cardinal of first order theories categorical in *one* uncountable cardinal [Mor65]. He introduced for that purpose an ordinal valued rank, later shown to be finite by J. Baldwin in the uncountably categorical context [Bal73], and this rank can be seen as an abstract version of the Zariski dimension in algebraic geometry over an algebraically closed field.

In particular, the category of groups of finite Morley rank encapsulates finite groups and algebraic groups over algebraically closed fields. One of the most basic tools for analyzing finite groups is Sylow theory, and in algebraic groups semisimplicity and unipotence theory play a similar role. It is thus not surprising to see these two theories, together with all

† Parts of this work were done while the authors were visiting the Isaac Newton Institute, Cambridge, during the model theory program in the spring 2005.

conjugacy results they suggest, having enormous and close developments in the more abstract category of groups of finite Morley rank. The present paper is intended to give an exhaustive survey on these parallel developments.

In a connected linear algebraic group, the centralizers of maximal tori are conjugate and cover the group generically. In the category of groups of finite Morley rank, these Cartan subgroups are best approximated by *Carter* subgroups, which are defined merely by the outstanding properties of being definable, connected, nilpotent, and of finite index in their normalizers. The main feature of Carter subgroups is their existence in any group of finite Morley rank. They constitute, together with all relevant approximations of semisimplicity and unipotence, the core of our preoccupations in this paper.

Sylow theory, as the study of maximal p-subgroups, is well understood for any p in *solvable* groups of finite Morley rank, and in any group of finite Morley rank for the prime $p = 2$. The second point is the key for a classification program of simple groups of finite Morley rank, suggested by A. Borovik and based on the architecture of the Classification of the Finite Simple Groups. In this process, some specific developments have naturally been needed for groups of finite Morley rank. In this context there is a priori no Jordan decomposition as in the linear algebraic context, and hence no nice distinction between semisimple and unipotent elements. The situation is furthermore enormously complicated by some so-called *bad* fields, as we will see in §1.7. Nevertheless, the finiteness of the Morley rank has allowed J. Burdges to develop a graduated notion of unipotence in this general context. This graduated notion of unipotence leads naturally to a new kind of Sylow theory, not related to torsion elements directly, but rather to the *unipotence degree* of the subgroups involved. In finite groups the study of Carter subgroups mostly boils down to Sylow theory; in groups of finite Morley rank this is replaced by this new kind of Sylow theory.

More precisely, we deal here with $\tilde{p}$-groups, where $\tilde{p} = (p, r)$ and p should be understood as the usual prime, or ∞ when dealing with elements of infinite order or merely divisible groups (which is more or less the same up to saturation). In this theory the unipotence degree r measures simultaneously how much a $\tilde{p}$-group can act on, and be acted upon by, another such group. Our $\tilde{p}$-groups are connected and nilpotent by definition and can really be thought of as the p-groups from finite group theory, incorporating the important unipotence degree parameter