ENTIRE HOLOMORPHIC MAPPINGS IN ONE AND SEVERAL COMPLEX VARIABLES

BY

PHILLIP A. GRIFFITHS

Hermann Weyl Lectures
The Institute for Advanced Study

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HERMANN WEYL LECTURES

The Hermann Weyl lectures are organized and sponsored by the School of Mathematics of the Institute for Advanced Study. Their aim is to provide broad surveys of various topics in mathematics, accessible to nonspecialists, to be eventually published in the Annals of Mathematics Studies.

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ARMAND BOREL
JOHN W. MILNOR

INDEX OF NOTATIONS

$\mathbf{C}^n$ is complex Euclidean space with coordinates $z = (z_1, \cdots, z_n)$;

$(z,w) = \sum_{i=1}^{n} z_i \bar{w}_i$ and $\|z\|^2 = (z,z)$;

$\mathbf{P}^n$ is complex projective space with homogeneous coordinates

$$Z = [z_0, \cdots, z_n];$$

$\mathbf{P}^1 = \mathbf{C} \cup \{\infty\}$ is the Riemannian sphere;

$B[r] = \{z \in \mathbf{C}^n : \|z\| < r\}$ is the ball of radius r in $\mathbf{C}^n$;

$S[r] = S \cap B[r]$ for any set $S \subset \mathbf{C}^n$;

$\Delta(r) = \{z \in \mathbf{C} : |z| < r\}$ is the disc of radius r in $\mathbf{C}$;

$\Delta = \Delta(1)$ is the unit disc;

$\Delta^* = \{\zeta \in \mathbf{C} : 0 < |\zeta| < 1\}$ is the punctured disc;

$\Delta^*(R) = \{z \in \mathbf{C}^n : |z_i| < R_i$ and $R = (R_1, \cdots, R_n)\}$ is a polycylinder in $\mathbf{C}^n$;

$\Delta^*_{k,n} = (\Delta^*)^k \times (\Delta)^{n-k}$ is a punctured polycylinder;

$\Phi, \Psi, \cdots$ denote volume forms;

$\omega, \phi, \psi, \eta, \cdots$ denote $(1,1)$ forms;

$d^c = \frac{\sqrt{-1}}{4\pi}(\bar{\partial} - \partial)$;

On $\mathbf{C}$ with $z = re^{i\theta}$,

$$d^c = \frac{1}{4\pi} r \frac{\partial}{\partial r} \otimes d\theta - \frac{1}{4\pi}\frac{1}{r}\frac{\partial}{\partial \theta} \otimes dr ;$$

$\phi = dd^c\|z\|^2 = \frac{\sqrt{-1}}{2\pi}\left(\sum_{i=1}^{n} dz_i \wedge d\bar{z}_c\right)$ is the standard Kähler form on $\mathbf{C}^n$;

$\omega = dd^c \log\|z\|^2$ is the pull-back to $\mathbf{C}^n - \{0\}$ of the Fubini-Study Kähler metric on $\mathbf{P}^{n-1}$;

A holomorphic line bundle is denoted by $L \to M$;

$H \to \mathbf{P}^n$ is the hyperplane line bundle;

D is a divisor and [D] the corresponding line bundle;

$c_1(L)$ is the Chern form (curvature form) of a Hermitian line bundle;

$\mathcal{O}(M,L)$ is the space of holomorphic sections of $L \to M$;

$|L|$ is the projective space of divisors of sections $s \in \mathcal{O}(M,L)$;

$H^2_{DR}(M,R)$ is the 2^{nd} deRham cohomology of M;

A real (1,1) form ψ on M is positive in case locally

$$\psi = \frac{\sqrt{-1}}{2} \sum \psi_{ij} dz_i \wedge d\bar{z}_j$$

where (ψ_{ij}) is a positive definite Hermitian matrix;

$[\psi]$ denotes the class in $H^2_{DR}(M,R)$ of a closed form ψ on M;

A class χ in $H^2_{DR}(M,R)$ is positive in case $\chi = [\psi]$ for some positive form ψ;

K_M is the canonical line bundle of M;

L^* is the dual line bundle to $L \to M$;

$\mathcal{O}(C^n)$ are the entire holomorphic functions on C^n.

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Entire Holomorphic Mappings in One and Several Complex Variables

ENTIRE HOLOMORPHIC MAPPINGS IN ONE AND SEVERAL COMPLEX VARIABLES

Phillip A. Griffiths*

INTRODUCTION

(a) *Some general remarks*

These talks will be concerned with the value distribution theory of an *entire holomorphic mapping*

$$f : \mathbf{C}^n \to M$$

where M is a compact, complex manifold. The theory began with R. Nevanlinna's quantitative refinement of the Picard theorem concerning a non-constant entire meromorphic function

$$f : \mathbf{C} \to \mathbf{P}^1 .$$

If we let $n_f(a, r)$ be the number of solutions to the equation

$$f(z) = a , \quad |z| < r \ \text{ and } \ a \in \mathbf{P}^1 ,$$

then Picard's theorem says that the sum

$$n_f(a, r) + n_f(b, r) + n_f(c, r)$$

is eventually positive, where a, b, c are three distinct points on $\mathbf{P}^1$. Roughly speaking, Nevanlinna's refinement states that the above sum is eventually larger than the average

$$t_f(r) = \int_{a \in \mathbf{P}^1} n_f(a, r)\, da$$

*This work was partially supported by NSF Grant GP38886.

number of solutions to the equation in question. Since the appearance of Nevanlinna's book [31], there has been considerable attention to the subject of value distribution theory, first in the classical case of an entire meromorphic function [25], then in the study of entire holomorphic curves in projective space ([44] and [45]), and, in recent years, in the general theory of holomorphic mappings between arbitrary complex manifolds (cf. [37] for a survey).

We shall concentrate on the *equidimensional case* where $\dim_{\mathbf{C}} M = n$ and where f is *non-degenerate* in the sense that the Jacobian determinant of f is non-identically zero. Aside from the Ahlfors theory of holomorphic curves in $\mathbf{P}^n$, it is here that the defect relations of R. Nevanlinna have been most directly generalized. For example, Picard's theorem becomes the assertion that no such f can omit a divisor on M which has simple normal crossings and whose Chern class is larger than that of the anti-canonical divisor on M. The corresponding defect relation will be proved in Chapter 3.

Aside from proving this theorem, there are two main purposes of these lectures. The first is to attempt to integrate more closely the deeper analytic aspects of the classical one variable theory with the formalism and algebro-geometric flavor in the several variable case. The second is to try to isolate the analytic concept of *growth* and differential-geometric notion of *negative curvature* as being perhaps most basic to the theory. A glance at the table of contents should make it pretty clear how our discussion has been centered around these two purposes.

One other aspect of these notes is that we have tried to give the heuristic reasoning which historically led to the recognition that growth and curvature were central to theory. Aside from the original proof of Picard's theorem using the modular function, we have included three additional proofs, among them the "elementary proof" of Emile Borel in which the central importance of growth was first clearly exhibited. Similarly, aside from the negative curvature derivation of the general defect relation, we have included the two classical proofs due to R. Nevanlinna and Ahlfors, each of which has its own distinct merit.

(b) *General references and background material*

The classic book in the subject is [31] by R. Nevanlinna. A second book of his [32] and the more recent monograph by Hayman [25] contain further discussion of the value distribution theory of an entire meromorphic function. All that is required to read any of these books is standard basic knowledge of complex function theory.

In several variables, we shall use the formalism of complex manifold theory, especially that centered around divisors, line bundles, and their Chern forms and subsequent Chern classes. The basic references here are the books by Chern [11] and Wells [43]. In practice all that we shall really require is fully explained in the introduction (pp. 151-155) of Griffiths-King [22].

From several complex variables, one needs to know a little about the local structure of an analytic hypersurface, especially as regards integration over an analytic hypersurface and Stokes' theorem in this situation. For the former the second chapter of Gunning and Rossi [24] is more than sufficient, and for integration we suggest the notes by Stolzenberg [41].

Aside from the facts that integration of a smooth differential form over a possibly singular analytic variety is possible and that Stokes' formula is valid, the fundamental result we shall use concerning integration on complex varieties is the *Wirtinger theorem*:

Let $ds^2 = \sum_{i,j} h_{ij}\, dz_i\, d\bar{z}_j$ *and* V *be respectively a Hermitian metric and* k*-dimensional analytic variety, both defined in some neighborhood of the closure of an open set* $U \subset \mathbb{C}^n$. *Let* $\phi = \frac{\sqrt{-1}}{2} \sum_{i,j} h_{ij}\, dz_i \wedge d\bar{z}_j$ *be the exterior* (1,1) *form associated to the Hermitian metric and* vol(V) *the* 2k*-dimensional volume of* $V \cap U$ *computed with respect to the Riemannian metric associated to* ds^2. *Then*

(0.1) $$\mathrm{vol}(V) = \frac{1}{k!} \int_{V \cap U} \phi^k$$

where $\phi^k = \underbrace{\phi \wedge \cdots \wedge \phi}_{\text{k-times}}$.

PROOF. If V^* is the set of non-singular points of V, then by definition $\mathrm{vol}(V) = \mathrm{vol}(V^*)$ and $\int_{V \cap U} \phi^k = \int_{V^* \cap U} \phi^k$. Thus it suffices to prove (0.1) in case V is a complex submanifold of U, and this obviously reduces to establishing (0.1) in case V = U. By applying the Gram-Schmidt process to the differentials $dz_1, \cdots, dz_n$ relative to the Hermitian metric (h_{ij}), we may write

$$ds^2 = \sum_{i=1}^{n} \phi_i \bar{\phi}_i$$

where the $\phi_i = \sum_j a_{ij} dz_j$ give a C^∞ basis for the (1,0) forms over U. Expanding

$$\omega_i = \alpha_i + \sqrt{-1}\, \beta_i$$

in real and imaginary parts, the associated Riemannian metric is

$$ds^2 = \sum_{i=1}^{n} \alpha_i^2 + \beta_i^2 .$$

The volume form for this metric is, by definition,

$$d\mu = \alpha_1 \wedge \beta_1 \wedge \cdots \wedge \alpha_n \wedge \beta_n .$$

On the other hand,

$$\phi = \frac{\sqrt{-1}}{2} \left(\sum_{i=1}^{n} \phi_i \wedge \bar{\phi}_i \right)$$

so that

$$\begin{aligned} \phi^n &= \left(\frac{\sqrt{-1}}{2}\right)^n n!\, \phi_1 \wedge \bar{\phi}_1 \wedge \cdots \wedge \phi_n \wedge \bar{\phi}_n \\ &= n!\, \alpha_1 \wedge \beta_1 \wedge \cdots \wedge \alpha_n \wedge \beta_n \\ &= n!\, d\mu . \end{aligned}$$

Q.E.D.

The principle that the volume of an analytic variety $V \subset \mathbf{C}^n$ is computed by integration of a differential form defined on all of $\mathbf{C}^n$, as opposed to the computation of arclength, surface area, etc. in the real case, is of fundamental importance.

CHAPTER 1
ORDERS OF GROWTH

(a) *Some heuristic comments*

In its simplest terms, the study of entire holomorphic mappings is concerned with *growth*. For example, when the well-known growth behavior of a polynomial is plugged into the Cauchy integral formula

$$n_p(0, r) = \frac{1}{2\pi\sqrt{-1}} \int_{|z|=r} \frac{p'(z)\,dz}{p(z)}$$

one obtains the fundamental theorem of algebra. Emile Borel realized that growth was also essential to an understanding of *Picard's theorem*, viewed as a transcendental analogue of the fundamental theorem of algebra. Here is the heuristic reasoning behind his proof of that theorem:

An entire holomorphic mapping $f: \mathbf{C} \to \mathbf{P}^1$ may be written in homogeneous coordinates as $f(z) = [f_0(z), f_1(z)]$ where f_0 and f_1 are entire holomorphic functions having no common zeroes. We assume that f is nonconstant and omits three points, which may be taken to be $[1, 0]$, $[0, 1]$, and $[1, -1]$. Then

$$\begin{cases} f_0 = e^{h_0}, \ f_1 = e^{h_1} \\ \text{and} \\ e^{h_0}/e^{h_1} + 1 = e^{h_2} \end{cases}$$

where h_0, h_1 and h_2 are entire holomorphic functions. Multiplying the second equation by e^{-h_2} gives a linear relation