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# Invariant Descriptive Set Theory

**Su Gao**

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*To Shuang, Alvin, and Tony, with love*

# Preface

My intention in writing this book is to bring into one place the basics of invariant descriptive set theory, also known as the descriptive set theory of definable equivalence relations. Invariant descriptive set theory has been an active field of research for about 20 years. Many researchers and students are impressed by its fast development and its relevance to other fields of mathematics, and would like to be better acquainted with the theory. I have tried to make this book as self-contained as possible, and at the same time covered what I believe to be the essential concepts, methods, and results.

The book is designed as a graduate text suitable for a year-long course. I have kept the sections short so that they can be used as lecture notes, and most of the sections are followed by a number of exercise problems. Many exercises are propositions and even theorems needed later in the book. So the student is urged to make a serious effort to work them out.

Ideally, the student should have some experience with classical and effective descriptive set theory before reading this book. But since this is most likely not the case, I have only assumed that the student knows some general topology. In the first chapter a review of classical and effective descriptive set theory is given, and throughout the book results are recalled as they become necessary. I can imagine that it is hard, but possible, for a student who has never seen any descriptive set theory to get started on the subject, but I believe that, with patience and diligence, the obstacles will be overcome eventually.

I have to remark, primarily for the experts in the field, that the book is not intended to be a comprehensive account of all aspects of invariant descriptive set theory. A reader who is familiar with the materials of the book and who is interested in further developments should have no problem following the current literature on many topics and applications. The selection of topics contained in this book was greatly influenced by the book of Becker and Kechris [8] and some unpublished notes of Kechris.

I would like to thank Julia Knight for inviting me to give a short course on invariant descriptive set theory at the University of Notre Dame in 2005. The first ideas for this book came from the notes for that short course. I would also like to thank the participants of the short course for typing the notes and for conversations on the topic. I am grateful to Dave Marker and Peter Cholak for the encouragement to write a graduate textbook on the subject. Special thanks are due to Longyun Ding and Vincent Kieftenbeld for comments and suggestions on the manuscript.

I would like to acknowledge the financial support of the National Science Foundation and the University of North Texas for the composition of this book and for related research. It would be impossible for me to write this book without the faculty developmental leave granted by the University of North Texas. Many thanks to all at CRC/Taylor & Francis for their untiring work to make this book a reality.

I am indebted to Greg Hjorth for leading me into the field and to Alekos Kechris for many years of advice and support. It is a privilege for me to be acquainted with many colleagues and experts in the field, too numerous to list (see the References and Index), whose research results shaped this book. I benefited a great deal from communications with them and from their contributions to the literature. I present this book to them with gratitude and pride.

Su Gao
Denton, Texas

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# Part I

# Polish Group Actions

# Chapter 1

# Preliminaries

This chapter reviews the concepts and results of classical and effective descriptive set theory that will be used in this book. Classical descriptive set theory was founded by Baire, Borel, Lebesgue, Luzin, Suslin, Sierpinski, and others in the first two decades of the twentieth century. The theory studies the *descriptive complexity* of sets of real numbers arising in ordinary mathematics, mostly in topology and analysis. The most striking achievements of this theory are the proofs of regularity properties of low-level definable sets of reals.

Effective descriptive set theory was created later in the century by introducing into the classical theory the new and powerful tools developed from recursion theory (now called computability theory). Computability theory came about from completely different motivations and was invented by another group of great minds such as Gödel, Church, Turing, Kleene, and others. It provided a framework to understand the structural and computational complexity of sets and functions (mostly pertaining to natural numbers). The classical theory is much better understood from the perspective of the effective theory.

In this chapter we review the basic concepts and results of both classical and effective theory that are relevant to the remainder of the book. Readers unfamiliar with these topics should be able to get a working idea about the content and the flavor of the theory by reading this chapter alone, and especially if they are diligent enough to work out the exercise problems in this chapter. Later in the book there will be more specific concepts and results being reviewed as they become necessary tools. In addition, some proofs are given in the appendix. The reader can probably get by with these reviews without ever systematically studying the classical and effective descriptive set theory, but an understanding of the comprehensive theory will be hugely advantageous.

For a complete treatment of these topics the reader can consult the standard references. For classical descriptive set theory the standard textbook is [97]

> A. S. Kechris, *Classical Descriptive Set Theory.* Graduate Texts in Mathematics 156. Springer-Verlag, New York, 1995.

And for effective descriptive set theory our standard source is [126]

> Y. N. Moschovakis, *Descriptive Set Theory.* Studies in Logic and

Foundations of Mathematics 100. North-Holland, Amsterdam, New York, 1980.

In addition, there are more concise treatments of the subject which provide efficient inroads to the theory and can be used as alternatives for or in conjunction with the standard texts: [148] by Srivastava, [125] by Martin and Kechris, [116] by Mansfield and Weitkamp, and [133] by Sacks. Thus in this chapter our review will be pragmatic and sketchy. Many facts, even theorems, are given without proof. Also we have left out important aspects of the theory just to get the reader quickly prepared to deal with the main topics of the book starting in Chapter 2.

Let the story begin.

---

## 1.1 Polish spaces

***Definition 1.1.1***
A topological space is **Polish** if it is separable and completely metrizable.

Some basic properties of Polish spaces are gathered in the following proposition.

***Proposition 1.1.2***

(a) *Any Polish space is second countable and normal.*

(b) *Any Polish space is Baire. (Recall that a topological space is* **Baire** *if the intersection of countable many dense open sets is dense.)*

(c) *A finite or countable product of Polish spaces is Polish.*

(d) *A subspace $Y$ of a Polish space $X$ is Polish iff $Y$ is $G_\delta$ in $X$, that is, $Y$ is the intersection of countably many open sets in $X$.*

(e) *A quotient space of a Polish space is not necessarily Polish.*

Clause (b) in the proposition is a direct consequence of the **Baire category theorem**, which says that a complete metric space is Baire. Examples of Polish spaces are abundant in mathematics. Some of the most familiar examples are listed below.

***Example 1.1.3***

(1) All countable spaces with the discrete topology are Polish. These include

the following spaces:

$$\mathbb{N} = \omega = \{0, 1, 2, \dots\},$$

$$\mathbb{N}_+ = \mathbb{N} - \{0\} = \{1, 2, \dots\},$$

$$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}.$$

Throughout this book we use $\mathbb{N}$ and $\omega$ interchangeably.

(2) $\mathbb{R}^n$ with the usual topology for $1 \leq n \leq \omega$ are Polish.

(3) The Baire space $\mathcal{N} = \omega^\omega$ is Polish. A complete metric on $\mathcal{N}$ is defined by

$$d(x, y) = \begin{cases} 0, & \text{if } x = y, \\ 2^{-n-1}, & \text{if } n \in \omega \text{ is the least such that } x(n) \neq y(n). \end{cases}$$

(4) The Cantor space $2^\omega$ is a closed subspace of $\mathcal{N}$, hence is Polish.

(5) All separable Banach spaces, such as $c_0$ and $\ell_p$ ($1 \leq p < \infty$), are Polish. Note that $\ell_\infty$ is not separable and therefore not a Polish space.

(6) All compact metrizable spaces are Polish (see Exercise 1.1.1).

Some nontrivial examples of Polish spaces involve hyperspaces of sets or functions. We examine two such examples below.

Let $d$ be a compatible metric on a Polish space $X$. Then we can define a compatible metric $d'$ on $X$ with the property that $d' \leq 1$ as follows:

$$d'(x, y) = \frac{d(x, y)}{1 + d(x, y)}.$$

Moreover, if $d$ is complete then so is $d'$. If $x \in X$ and $A \subseteq X$, then we denote

$$d(x, A) = d(A, x) = \inf\{d(x, y) : y \in A\}.$$

Let $X$ be a Polish space. Let $K(X)$ denote the space of all compact subsets of $X$ equipped with the **Vietoris topology** generated by subbasic open sets of the following form:

$$\{K \in K(X) : K \subseteq U\}, \text{ or}$$

$$\{K \in K(X) : K \cap U \neq \emptyset\},$$

for $U$ open in $X$. Then $K(X)$ is Polish. An explicit compatible metric on $K(X)$ is known as the **Hausdorff metric**. To define it, let $d$ be a compatible