Rüdiger Seydel

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Practical Bifurcation and Stability Analysis

Third Edition

Springer

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Rüdiger Seydel
Universität zu Köln
Mathematisches Institut
Weyertal 86-90
50931 Köln
Germany
seydel@math.uni-koeln.de

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Preface

Fifteen years have elapsed after the second edition of *Practical Bifurcation and Stability Analysis* was published. During that time period the field of computational bifurcation has become mature. Today, bifurcation mechanisms are widely accepted as decisive phenomena for explaining and understanding stability and structural change. Along with the high level of sophistication that bifurcation analysis has reached, the research on basic computational bifurcation algorithms is essentially completed, at least in ordinary differential equations. The focus has been shifting from mathematical foundations towards applications.

The evolution *from equilibrium to chaos* has become commonplace and is no longer at the cutting edge of innovation. But the corresponding methods of *practical bifurcation and stability analysis* remain indispensable instruments in all applications of mathematics. This constant need for practical bifurcation and stability analysis has stimulated an effort to maintain this book on a present-day level. The author's endeavor has resulted in this third edition. It is based on more than three decades of practical experience with the subject, and on many courses given at several universities.

Like the previous editions, this third edition consists of three parts. In the first part (Chapters 1 to 3) an introduction into bifurcation and stability phenomena is given, basically restricted to models built of ordinary differential equations. Phenomena such as birth of limit cycles, hysteresis, or period doubling are explained. The second part (Chapters 4 to 7) introduces computational methods for analyzing bifurcation and stability. This includes continuation and branch switching as basic means. The final part (Chapters 8 and 9) gives qualitative insight that may help in understanding and assessing computational results. Such an interpretation of numerical results is based on singularity theory, catastrophe theory, and chaos theory.

This book emphasizes basic principles and shows the reader how the methods result from combining and, on occasion, modifying the underlying principles. The book is written to address the needs of scientists and engineers and to attract mathematicians. Mathematical formalism is kept to a minimum; the style is not technical, and is often motivating rather than proving. Compelling examples and geometrical interpretations are essential ingredients in the style. Exercises and projects complete the text. The book is self-contained, assuming only basic knowledge in calculus. The extensive bibliography includes many references on analytical and numerical methods, applications in science and engineering, and software. The references may serve as first steps in finding additional material for further research.

The book attempts to provide a practical guide for the performance of parameter studies.

New in This Edition

This third edition has been partly reorganized. The main change is a newly written Chapter 3. The third chapter of the second edition was removed, part of its contents was added to the fourth chapter. The new Chapter 3 is devoted to applications and extensions of standard ODE approaches. It includes brief expositions on delay differential equations, on differential-algebraic equations, and on pattern formation. This last aspect is concentrating on reaction-diffusion problems with applications in nerve models. Finally, this new third chapter addresses the aspect of deterministic risk, which can be tied to bifurcation. Applications include production of blood cells, dry friction, a flip-flop circuit, Turing bifurcation, and an electric power generator.

In addition to the new Chapter 3, several new sections have been inserted. In Chapter 5, the new Section 5.5 summarizes the information on second-order derivatives. In Chapter 7, on periodic orbits, the Section 7.6 on numerical aspects of bifurcation was enlarged. In Chapter 9, the section on fractal dimensions has been extended, and a new section has been added on the control of chaos, with focus on the OGY method.

Apart from these expanded sections, the entire book has been thoroughly reworked and revised. There are many new figures, while other figures have been improved. A considerable number of new references guide the reader to some more recent research or applications. The additions of this third edition are substantial; this may be quantified by the increase in the number of pages (+16%), figures (+19%), or references (+22%). The author has attempted to follow the now generally adopted practice to use *branching* and *bifurcation* as synonyms.

How to Use the Book

A path is outlined listing those sections that provide the general introduction into bifurcation and stability. Readers without urgent interest in computational aspects may wish to concentrate on the following:

Sections 1.1 to 1.4;
all of Chapter 2;
part of Chapter 3;
part of Section 5.4.2, and Sections 5.5, 5.6.4, and 5.6.5;
Section 6.1, example in Section 6.4, and Section 6.8;

Sections 7.1 to 7.4, 7.7, and 7.8;
all of Chapter 8; and
all of Chapter 9.

Additional information and less important remarks are set in small print. On first reading, the reader may skip these parts without harm. Readers with little mathematical background are encouraged to read Appendices 1 to 3 first. Solutions to several of the exercises are given later in the text. References are not meant as required reading, but are hints to help those readers interested in further study. The figures framed in boxes are immediate output of numerical software.

I hope that this book inspires readers to perform their own experimental studies. The many examples and figures should provide a basis and motivation to start right away.

Köln, September 2009 *Rüdiger Seydel*

Notation

Problem-Inherent Variables

λ	scalar parameter to be varied (bifurcation parameter)
$\mathbf{y}$	vector of state variables, vector function, solution of an equation
n	number of components of vectors $\mathbf{y}$ or $\mathbf{f}$
$\mathbf{f}$	vector function, defines the dynamics of the problem that is to be solved; typical equation $\mathbf{f}(\mathbf{y}, \lambda) = \mathbf{0}$
t	independent variable, often time
$\dot{\mathbf{y}}$	derivative of $\mathbf{y}$ with respect to time, $\dot{\mathbf{y}} = \mathrm{d}\mathbf{y}/\mathrm{d}t$
$\mathbf{y}'$	derivative of $\mathbf{y}$ with respect to a general independent variable
a, b	define an interval in which t varies, $a \leq t \leq b$
x	spatial variable, may be scalar or vector with up to three components
$\mathbf{r}$	vector function, often used to define boundary conditions as in $\mathbf{r}(\mathbf{y}(a), \mathbf{y}(b)) = \mathbf{0}$
T	period in case of a periodic oscillation
γ	additional scalar parameter

Notations for a General Analysis

In particular examples, several of the following meanings are sometimes superseded by a local meaning

Specific Versions of $\mathbf{y}$ and $\boldsymbol{\lambda}$

λ_0	specific parameter value of a bifurcation point
$\mathbf{y}_0$	specific n-vector of a bifurcation point
y_i	ith component of vector $\mathbf{y}$
$\mathbf{y}^j$	jth continuation step (j is not an exponent here), specific solution
λ_j	specific parameter value, corresponds to $\mathbf{y}^j$
$\mathbf{y}^{(\nu)}$	iterates of a map. For example, Newton iterate; for $\nu = 1, 2, \ldots$ sequence of vectors converging to a solution $\mathbf{y}$
$\mathbf{y}^{\mathrm{s}}$	stationary solution ($\mathbf{y}_{\mathrm{s}}$ in Sections 6.6 and 6.7)

Integers

k frequently, the kth component has a special meaning
N number of nodes of a discretization
$i,\ j,\ l, m, \nu$ other integers
(Note that i denotes the imaginary unit.)

Scalars

$[\mathbf{y}]$ a scalar measure of $\mathbf{y}$ (cf. Section 2.2)
ρ radius
ϑ angle
ω frequency
ϵ accuracy, error tolerance
δ distance between two solutions, or parameter
η value of a particular boundary condition
τ test function indicating bifurcation
Δ increment or decrement, sometimes acting as operator on the following variable; for instance, $\Delta\lambda$ means an increment in λ
s arclength
$u,\ v$ functions, often solutions of scalar differential equations
σ step length
p parameterization, or phase condition, or polynomial
c_i constants
μ $= \alpha + \mathrm{i}\beta$ complex-conjugate eigenvalue
$\zeta,\ \xi$ further scalars with local meaning

Vectors

$\mathbf{z}$ n-vector (column) in various roles, as tangent, or initial vector of a trajectory, or emanating solution, or eigenvector
$\mathbf{z}^{tr}$ row vector (transposed column)
$\mathbf{d}$ difference between two n-vectors
$\mathbf{h}$ n-vector, solution of a linearization; $\mathbf{h}_0$ or $\bar{\mathbf{h}}$ are related n-vectors
$\mathbf{e}_i$ ith unit vector (cf. Appendix A.2)
φ $\varphi(t; \mathbf{z})$ is the trajectory starting at $\mathbf{z}$ (Eq. (7.7))
$\mathbf{w}$ eigenvector, also $\mathbf{w}^k$
μ $= \alpha + \mathrm{i}\beta$ vector of eigenvalues
Λ vector of parameters
$\mathbf{Y}$ vector with more than n components, contains $\mathbf{y}$ as subvector
$\mathbf{F}$ vector with more than n components, contains $\mathbf{f}$ as subvector
$\mathbf{R}$ vector with more than n components, contains $\mathbf{r}$ as subvector
$\mathbf{P}$ map, Poincaré map
$\mathbf{q}$ argument of Poincaré map

n^2-Matrices (n rows, n columns)

$\mathbf{I}$ identity matrix
$\mathbf{J}$ Jacobian matrix $\mathbf{f_y}$ of first-order partial derivatives of $\mathbf{f}(\mathbf{y})$ w.r.t. $\mathbf{y}$
$\mathbf{M}$ monodromy matrix
$\mathbf{A}, \mathbf{B}$ derivatives of boundary conditions (Eq. (6.12))
$\mathbf{E}, \mathbf{G}_j$ special matrices of multiple shooting (Eq. (6.21), Eq. (6.22))
Φ, $\mathbf{Z}$ fundamental solution matrices (cf. Section 7.2)
$\mathbf{S}$ element of a group $\mathcal{G}$

Further Notations

Ω hypersurface
$\mathcal{M}$ manifold
$\mathcal{G}$ group, see Appendix A.7
$\in$ "in," element of a set, $\notin$ for "not in"
^{tr} as superscript means "transposed"
ln natural logarithm
Re real part
Im imaginary part
∂ partial derivative
$\bar{\mathbf{y}}, \bar{\lambda}, \bar{\mathbf{h}}, \bar{\mathbf{z}}$ overbar characterizes approximations
∇u gradient of u (∇ is the "del" operator)
$\nabla^2 u$ Laplacian operator (summation of second-order derivatives)
$\nabla \cdot u$ divergence of u
$:=$ defining equation; the left side is "new" and is defined by the right-hand side; see, for example, Eq. (4.14)
$O(\sigma)$ terms of order of σ
$\| \ \|$ vector norm, see Appendix A.1

Abbreviations

t.h.o. terms of higher order
w.r.t. with respect to
DAE differential-algebraic equation (cf. Section 3.3)
ODE ordinary differential equation
OGY Ott-Grebogi-Yorke method (cf. Section 9.6)
PDE partial differential equation
UPO unstable periodic orbit

Contents

1 Introduction and Prerequisites

1.1 A Nonmathematical Introduction

Every day of our lives we experience changes that occur either gradually or suddenly. We often characterize these changes as quantitative or qualitative, respectively. For example, consider the following simple experiment (Figure 1.1). Imagine a board supported at both ends, with a load on top. If the load λ is not too large, the board will take a bent shape with a deformation depending on the magnitude of λ and on the board's material properties (such as stiffness, K). This state of the board will remain *stable* in the sense that a small variation in the load λ (or in the stiffness K) leads to a state that is only slightly perturbed. Such a variation (described by Hooke's law) would be referred to as a quantitative change. The board is deformed within its elastic regime and will return to its original shape when the perturbation in λ is removed.

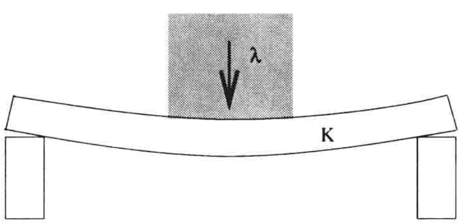

Fig. 1.1. Bending of a board

The situation changes abruptly when the load λ is increased beyond a certain *critical level* λ_0 at which the board breaks (Figure 1.2b). This sudden action is an example of a qualitative change; it will also take place when the material properties are changed beyond a certain limit (see Figure 1.2a). Suppose the shape of the board is modeled by some function (solution of an equation). Loosely speaking, we may say that there is a solution for load values $\lambda < \lambda_0$ and that this solution ceases to exist for $\lambda > \lambda_0$. The load λ and stiffness K are examples of *parameters*. The outcome of any experiment, any event, and any construction is controlled by parameters. The practical problem is to *control the state* of a system—that is, to find parameters such

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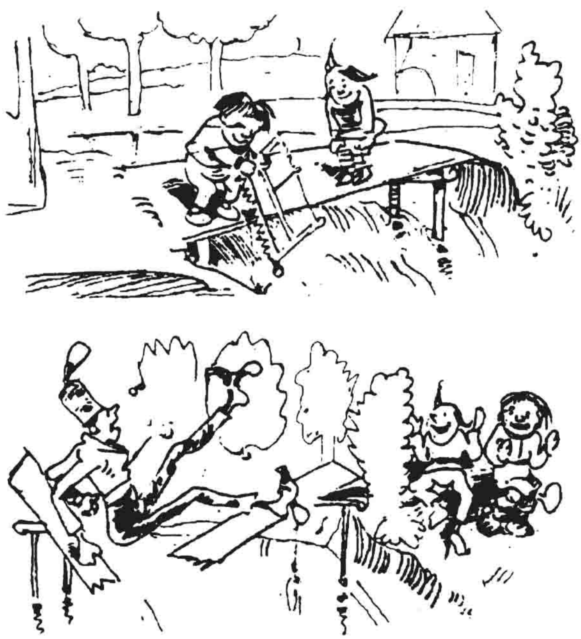
that the state fulfills our requirements. This role of parameters is occasionally emphasized by terms such as *control parameter*, or *design parameter*. Varying a parameter can result in a transition from a quantitative change to a qualitative change. The following pairs of verbs may serve as illustrations:

$$\begin{aligned} \text{bend} &\rightarrow \text{break} \\ \text{incline} &\rightarrow \text{tilt over} \\ \text{stretch} &\rightarrow \text{tear} \\ \text{inflate} &\rightarrow \text{burst}\,. \end{aligned}$$

The verbs on the left side stand for states that are stable under small perturbations; the response of each system is a quantitative one. This behavior ends abruptly at certain critical values of underlying parameters. The related drastic and irreversible change is reflected by the verbs on the right side. Close to a critical threshold the system becomes most sensitive; tiny perturbations may trigger drastic changes. To control a system may mean to find parameters such that the state of the system is safe from being close to a critical threshold. Since reaching a critical threshold often is considered a failure, the control of parameters is a central part of risk control.



(a)

(b)

Fig. 1.2. From W. Busch [Bus62]. After the original hand drawing in Wilhelm-Busch-Museum, Hannover