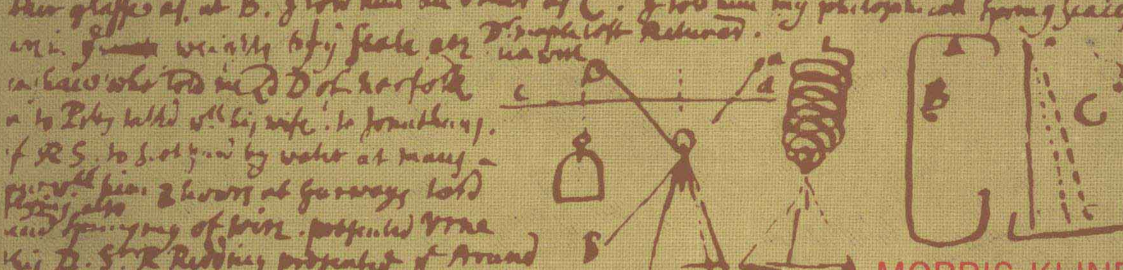
MATHEMATICS FOR LIBERAL ARTS



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PREFACE

"... *I consider that without understanding as much of the abstruser part of geometry, as Archimedes or Apollonius, one may understand enough to be assisted by it in the contemplation of nature; and that one needs not know the profoundest mysteries of it to be able to discern its usefulness. . . . I have often wished that I had employed about the speculative part of geometry, and the cultivation of the specious [symbolic] algebra I had been taught very young, a good part of that time and industry that I spent about surveying and fortification. . . .*"

ROBERT BOYLE

I believe as firmly as I have in the past that a mathematics course addressed to liberal arts students must present the scientific and humanistic import of the subject. Whereas mathematics proper makes little appeal and seems even less pointed to most of these students, the subject becomes highly significant to them when it is presented in a cultural context. In fact, the branches of elementary mathematics were created primarily to serve extra-mathematical needs and interests. In the very act of meeting such needs each of these creations has proved to have inestimable importance for man's understanding of the nature of his world and himself.

That so many professors have chosen to teach mathematics as an integral part of Western culture, as evidenced by their reception of my earlier book, *Mathematics: A Cultural Approach*, has been extremely gratifying. That book will continue to be available. In the present revision and abridgment, which has been designed to meet the needs of particular groups of students, the spirit of the original text has been preserved. The historical approach has been retained because it is intrinsically interesting, provides motivation for the introduction of various topics, and gives coherence to the body of material. Each topic or branch of mathematics dealt with is shown to be a response to human interests, and the cultural import of the technical development is presented. I adhered to the principle that the level of rigor should be suited to the mathematical age of the student rather than to the age of mathematics.

As in the earlier text, several of the topics are treated quite differently from what is now fashionable. These are the real number system, logic, and set theory. I tried to present these topics in a context and with a level of emphasis which I believe to be appropriate for an elementary course in mathematics. In this book, the axiomatic approach to the real numbers is formulated after the various types of numbers and their properties are derived from physical situations and uses. The treatment of logic is confined to the fundamentals of Aristotelian logic. And set theory serves as an illustration of a different kind of algebra.

The changes made in this revision are intended to suit special groups. Some students need more review and drill on elementary concepts and techniques than the earlier book provides. Others, chiefly those preparing for teaching on the elementary level, need to learn more about elementary mathematics than their high school courses covered. Teachers of twelfth-year high school courses and one-semester college courses often found the extensive amount of material in *Mathematics: A Cultural Approach* rather disconcerting because it offered so much more than could be covered.

To meet the needs of these groups I have made the following changes:

1. Four of the chapters devoted entirely to cultural influences have been dropped. The size of the original book has thereby been reduced considerably.
2. A few applications of mathematics to science have been omitted, primarily to reduce the size of the text.
3. Some of the chapters on technical topics, Chapter 3 on logic and mathematics, Chapter 4 on number, Chapter 5 on elementary algebra, and Chapter 21 on arithmetics and their algebras have been expanded.
4. Additional drill exercises have been added within a few chapters, and a set of review exercises providing practice in technique has been added to each of a number of chapters.
5. Improvements in presentation have been made in a number of places.

With respect to use in courses, it is probably true of the present text, as it is of the earlier one, that it contains more material than can be covered in some courses. However, many of the chapters as well as sections in chapters are not essential to the logical continuity. These chapters and sections have been starred (✱). Thus Chapter 10 on painting shows historically how mathematicians were led to projective geometry (Chapter 11), but from a logical standpoint, Chapter 10 is not needed in order to understand the succeeding chapter. Chapter 19 on musical sounds is an application of the material on the trigonometric functions in Chapter 18 but is not essential to the continuity. The two chapters on the calculus are not used in the succeeding chapters. Desirable as it may be to give students some idea of what the calculus is about, it may still be necessary in some classes to omit these chapters. The same can be said of the chapters on statistics (Chapter 22) and probability (Chapter 23).

As for sections within chapters, Chapter 6 on Euclidean geometry may well serve as an illustration. The mathematical material of this chapter is intended as a review of some basic ideas and theorems of Euclidean geometry and as an introduction to the conic sections. Some of the familiar applications are given in Section 6–3 (see the Table of Contents) and probably should be taken up. However the applications to light in Sections 6–4 and 6–6 and the discussion of cultural influences in Section 6–7 can be omitted.

Some of the material, whether or not included in the following recommendations for particular groups, can be left to student reading. In fact, the first two chapters were deliberately fashioned so that they could be *read* by students. The objective here, in addition to presenting intrinsically important ideas, was to induce students to read a mathematics book, to give them the confidence to do so, and to get them into the habit of doing so. It seems necessary to counter the students' impression, resulting no doubt from elementary and high school instruction in mathematics, that whereas history texts are to be read, mathematics texts are essentially reference books for formulas and homework exercises.

For courses emphasizing the number concept and its extension to algebra, it is possible to take advantage of the logical independence of numerous chapters and use Chapters 3 through 5 on reasoning, arithmetic, and algebra and Chapter 21 on different algebras. To pursue the development of this theme into the area of functions one can include Chapters 13 and 15.

Courses emphasizing geometry can utilize Chapters 6, 7, 11, 12, and 20 on Euclidean geometry, trigonometry, projective geometry, coordinate geometry, and non-Euclidean geometry respectively. Some algebra, that reviewed in Chapter 5, is involved in Chapters 7 and 12. If knowledge of the material of Chapter 5 cannot be presupposed, this chapter must precede the treatment of geometry.

The essence of the two preceding suggestions may be diagrammed thus:

Chapters 3, 4, 5

↓ ↓

Algebra — Chapters 21, 13, 15

Geometry — Chapters 6, 7, 11, 12, 20

Of course, starred sections in these chapters are optional.

For a one-semester liberal arts course, the basic content can be as follows:

Chapter 2	on a historical orientation,
Chapter 3	on logic and mathematics,
Chapters 4 and 5	on the number system and elementary algebra,
Chapter 6	through Section 6–5, on Euclidean geometry,
Chapter 7	through Section 7–3, on trigonometry,
Chapter 12	on coordinate geometry,
Chapter 13	on functions and their uses,
Chapter 14	through Section 14–4, on parametric equations,
Chapter 15	through Section 15–10, on the further use of functions in science,
Chapter 20	on non-Euclidean geometry,
Chapter 21	on different algebras.

Any additional material would enrich the course but would not be needed for continuity.

Though the teacher's problem of presenting material outside the domain of mathematics proper is far simpler with this text than with the earlier one, it may still be necessary to assure him that he need not hesitate to undertake this task. The feeling that one must be an authority in a subject to say anything about it is unfounded. We are all laymen outside the field of our own specialty, and we should not be ashamed to point this out to students. In contiguous areas we are merely giving indications of ideas that students may pursue further in other courses or in independent reading.

I hope that this text will serve the needs of the groups of students to which it is addressed and that, despite the somewhat greater emphasis on technical matters, it will convey the rich significance of mathematics.

I wish to thank my wife Helen for her critical scrutiny of the contents and her conscientious reading of the proofs. I wish to express, also, my thanks to members of the Addison-Wesley staff for very helpful suggestions and for their continuing support of a culturally oriented approach to mathematics.

New York, 1967 M.K.

CONTENTS

WHY MATHEMATICS?

In mathematics I can report no deficience, except it be that men do not sufficiently understand the excellent use of the Pure Mathematics. . . .

FRANCIS BACON

One can wisely doubt whether the study of mathematics is worth while and can find good authority to support him. As far back as about the year 400 A.D., St. Augustine, Bishop of Hippo in Africa and one of the great fathers of Christianity, had this to say:

> *The good Christian should beware of mathematicians and all those who make empty prophecies. The danger already exists that the mathematicians have made a covenant with the devil to darken the spirit and to confine man in the bonds of Hell.*

Perhaps St. Augustine, with prophetic insight into the conflicts which were to arise later between the mathematically minded scientists of recent centuries and religious leaders, was seeking to discourage the further development of the subject. At any rate there is no question as to his attitude.

At about the same time that St. Augustine lived, the Roman jurists ruled, under the Code of Mathematicians and Evil-Doers, that "to learn the art of geometry and to take part in public exercises, an art as damnable as mathematics, are forbidden."

Even the distinguished seventeenth-century contributor to mathematics, Blaise Pascal, decided after studying mankind that the pure sciences were not suited to it. In a letter to Fermat written on August 10, 1660, Pascal says: "To speak freely of mathematics, I find it the highest exercise of the spirit; but at the same time I know that it is so useless that I make little distinction between a man who is only a mathematician and a common artisan. Also, I call it the most beautiful profession in the world; but it is only a profession; and I have often said that it is good to make the attempt [to study mathematics], but not to use our forces: so that I would not take two steps for mathematics, and I am confident that you are strongly of my opinion." Pascal's famous injunction was, "Humble thyself, impotent reason."

The philosopher Arthur Schopenhauer, who despised mathematics, said many nasty things about the subject, among others that the lowest activity of the spirit is arithmetic, as is shown by the fact that it can be performed by a machine. Many other great men, for example, the poet Johann Wolfgang Goethe and the historian Edward Gibbon, have felt likewise and have not hesitated to express themselves. And so the student who dislikes the subject can claim to be in good, if not living, company.

In view of the support he can muster from authorities, the student may well inquire why he is asked to learn mathematics. Is it because Plato, some 2300 years ago, advocated mathematics to train the mind for philosophy? Is it because the Church in medieval times taught mathematics as a preparation for theological reasoning? Or is it because the commercial, industrial, and scientific life of the Western world needs mathematics so much? Perhaps the subject got into the curriculum by mistake, and no one has taken the trouble to throw it out. Certainly the student is justified in asking his teacher the very question which Mephistopheles put to Faust:

Is it right, I ask, is it even prudence,
To bore thyself and bore the students?

Perhaps we should begin our answers to these questions by pointing out that the men we cited as disliking or disapproving of mathematics were really exceptional. In the great periods of culture which preceded the present one, almost all educated people valued mathematics. The Greeks, who created the modern concept of mathematics, spoke unequivocally for its importance. During the Middle Ages and in the Renaissance, mathematics was never challenged as one of the most important studies. The seventeenth century was aglow not only with mathematical activity but with popular interest in the subject. We have the instance of Samuel Pepys, so much attracted by the rapidly expanding influence of mathematics that at the age of thirty he could no longer tolerate his own ignorance and begged to learn the subject. He began, incidentally, with the multiplication table, which he subsequently taught to his wife. In 1681 Pepys was elected president of the Royal Society, a post later held by Isaac Newton.

In perusing eighteenth-century literature, one is struck by the fact that the journals which were on the level of our *Harper's* and the *Atlantic Monthly* contained mathematical articles side by side with literary articles. The educated man and woman of the eighteenth century knew the mathematics of their day, felt obliged to be *au courant* with all important scientific developments, and read articles on them much as modern man reads articles on politics. These people were as much at home with Newton's mathematics and physics as with Pope's poetry.

The vastly increased importance of mathematics in our time makes it all the more imperative that the modern person know something of the nature and role of mathematics. It is true that the role of mathematics in our civilization is not always obvious, and the deeper and more complex modern applications are not readily comprehended even by specialists. But the essential nature and accomplishments of the subject can still be understood.

Perhaps we can see more easily why one should study mathematics if we take a moment to consider what mathematics is. Unfortunately the answer cannot be given in a single sentence or a single chapter. The subject has many facets or, some might say, is Hydra-headed. One can look at mathematics as a language, as a particular kind of logical structure, as a body of knowledge about number and space, as a series of methods for deriving conclusions, as the essence of our knowledge of the physical world, or merely as an amusing intellectual activity. Each of these features would in itself be difficult to describe accurately in a brief space.

Because it is impossible to give a concise and readily understandable definition of mathematics, some writers have suggested, rather evasively, that mathematics is what mathematicians do. But mathematicians are human beings, and most of the things they do are uninteresting and some, embarrassing to relate. The only merit in this proposed definition of mathematics is that it points up the fact that mathematics is a human creation.

A variation on the above definition which promises more help in understanding the nature, content, and values of mathematics, is that mathematics is what *mathematics does*. If we examine mathematics from the standpoint of what it is intended to and does accomplish, we shall undoubtedly gain a truer and clearer picture of the subject.

Mathematics is concerned primarily with what can be accomplished by reasoning. And here we face the first hurdle. Why should one reason? It is not a natural activity for the human animal. It is clear that one does not need reasoning to learn how to eat or to discover what foods maintain life. Man knew how to feed, clothe, and house himself millenniums before mathematics existed. Getting along with the opposite sex is an art rather than a science mastered by reasoning. One can engage in a multitude of occupations and even climb high in the business and industrial world without much use of reasoning and certainly without mathematics. One's social position is hardly elevated by a display of his knowledge of trigonometry. In fact, civilizations in which reasoning and mathematics played no role have endured and even flourished. If one were willing to reason, he could readily supply evidence to prove that reasoning is a dispensable activity.

Those who are opposed to reasoning will readily point out other methods of obtaining knowledge. Most people are in fact convinced that their senses are really more than adequate. The very common assertion "seeing is believing" expresses the common reliance upon the senses. But everyone should recognize that the senses are limited and often fallible and, even where accurate, must be interpreted. Let us consider, as an example, the sense of sight.