FOUNDATIONS FOR FINANCIAL ECONOMICS

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PREFACE

This book evolved from lecture notes we have used to teach introductory PhD courses in financial economics at the Massachusetts Institute of Technology, Stanford University, and the University of Pennsylvania. Its purpose is to provide the foundations for the study of modern financial economics. Rather than giving a superficial coverage of a wide range of topics, we have chosen to concentrate our discussion on individuals' consumption and portfolio decisions under uncertainty and their implications for the valuation of securities.

Chapters 1 through 6 discuss two–period models, where the consumption and portfolio decisions are made only once at the initial date of the economies. Chapter 1 analyzes an individual's behavior under uncertainty. This chapter also shows the comparative statics of an individual's optimal portfolio choice in an economy with one riskless and one risky asset when his initial wealth or attitude toward risk changes. Moreover, we provide sufficient conditions for these comparative statics to apply to economies with three or more assets. In Chapter 2, we discuss three concepts of ***stochastic dominance.*** The concepts of stochastic dominance identify conditions that allow risky assets to be ranked based on limited knowledge of individuals' preferences. Chapter 3 shows mathematical properties of a portfolio frontier – the collection of portfolios that have the minimum variance for different levels of expected rates of return. In Chapter 4, we give distributional conditions on the rates of return on assets so that individuals will optimally choose to hold portfolios on the portfolio frontier. As a consequence, expected rate of return on an asset is linearly related to its *beta*, which measures the contribution of the asset to the risk of a well-diversified portfolio. This is the *Capital Asset Pricing Model*. We also discuss in this chapter the *Arbitrage Pricing Theory*, which relates the expected rate of return of an asset to a number of *random factors*.

Chapter 5 begins our description of a state contingent security and its equilibrium valuation. A state contingent security pays one unit of consumption in one state of nature and nothing otherwise. Markets are said to be complete if there is a state contingent security for every state of nature. An allocation of consumption among individuals is said to be a Pareto optimal allocation if there is no other allocation that increases an individual's satisfaction without decreasing some other individual's satisfaction. We show how Pareto optimal allocations can be achieved in complete markets as well as in various other market structures. The allocational role of options, in particular, is demonstrated. This chapter also provides the necessary and sufficient conditions on individuals' utility functions for all Pareto optimal allocations to be achievable by holding the portfolio of all assets and borrowing or lending and discusses the relationship between these conditions and an aggregation result in securities markets. In Chapter 6, we present general pricing rules for securities that pay off in more than one state of nature and specialize these rules with additional preference restrictions. In particular, we derive a closed form solution for a call option written on a common stock when the random payoffs of the common stock and the aggregate consumption are jointly lognormally distributed and individuals' preferences are represented by power functions with the same exponent.

We discuss in Chapter 7 how multiperiod dynamic economies can be modeled. A multiperiod dynamic economy differs from a two–period static economy in that trading can take place at more than one date and individuals' expectations about future prices are therefore essential in an equilibrium specification. This leads to a notion of a rational expectations equilibrium. The general equilibrium valuation principles in a multiperiod dynamic economy are essentially the same as those in a two–period static economy. An important feature of a multiperiod economy demonstrated in detail is that a Pareto optimal allocation can be achieved by trading dynamically in a limited number of *long–lived securities*. Chapter 8 continues our discussion of a multiperiod economy with emphasis on valuation by arbitrage. We show the connection between an arbitrage–free price system and martingales. This connection allows us to compute prices of a derivative security in a simple way when the derivative security can be priced by arbitrage. As an example, we price a call option written on a stock when the stock price follows a binomial random walk.

A common feature of the economies in Chapters 1 through 8 is that individuals are endowed with the same information. In Chapter 9 we discuss economies in which individuals have differential information. We demonstrate that equilibrium properties can be very different from those in economies without differential information. Chapter 10 examines econometric issues of testing the Capital Asset Pricing Model. Some test statistics are given geometric interpretations in the context of Chapter 3.

Applications of information economics to financial markets have gained significant importance in recent years. Our coverage in Chapter 9 is limited in scope. Chapter 10 concentrates on econometric issues in testing the Capital Asset Pricing Model. Empirical aspects of many other theories developed in this book also deserve attention. Separate books can be written on the general topic areas of Chapters 9 and 10. Our selection of subjects covered in these two chapters is intended to be an introduction.

Besides providing material for introductory PhD courses in financial economics, this book can be used for a graduate/advanced undergraduate course in the economics of uncertainty. When supplemented with articles, this book can form the basis for a two semester course. Chapters 1 through 4 and Chapter 10 are recommended for the first semester, while Chapters 5 through 9 are recommended for the second semester. Although the level of presentation is rigorous in general, the necessary prerequisites are only intermediate level microeconomics, introductory econometrics, matrix algebra, and elementary calculus.

We owe a substantial debt to our academic colleagues who have contributed to the strong theoretical foundations of asset choice and valuation under uncertainty and to the empirical methodology for examining testable implications of the theory. This book presents and interprets materials in the existing literature and does not make original contributions of its own. In the end of each chapter, we try to give a brief attribution of the materials covered. But, undoubtedly, our attribution cannot be complete. Many of our colleagues provided either helpful comments on early drafts or encouragement throughout the years that this book was under preparation. Among them, special thanks go to Sudipto Bhattacharya and John Cox, from whom we have received continuous encouragement and invaluable suggestions on the selection of topics; and to Michael Gibbons and Craig MacKinlay, who have helped clarify some questions we had on the materials in Chapter 10. Many of our students have provided helpful comments and suggestions. Among them, special appreciations go to Ayman Hindy, who read through every chapter in detail and pointed out numerous mistakes in notation and derivations; to Caterina Nelsen, whose editorial help has proved indispensible; and to Ajay Dravid and Tomas Philipson, who gave helpful comments in terms of style and topic selection.

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CHAPTER 1
PREFERENCES REPRESENTATION AND RISK AVERSION

1.1. As we mentioned in the preface, the main focus of this book is on individuals' consumption and investment decisions under uncertainty and their implications for the valuation of securities. Individuals' consumption and investment decisions under uncertainty are undoubtedly influenced by many considerations. A commonly accepted theory of asset choice under uncertainty that provides the underpinnings for the analysis of asset demands uses the *expected utility hypothesis*. Under this hypothesis, each individual's consumption and investment decision is characterized as if he determines the probabilities of possible asset payoffs, assigns an index to each possible consumption outcome, and chooses the consumption and investment policy to maximize the expected value of the index. More formally, an individual's preferences have an *expected utility representation* if there exists a function u such that random consumption $\tilde{x}$ is preferred to random consumption $\tilde{y}$ if and only if

$$E[u(\tilde{x})] \geq E[u(\tilde{y})],$$

where $E[\cdot]$ is the expectation under the individual's probability belief.

In the first half of this chapter we give behavioral conditions that are necessary and sufficient for an individual's preferences to have an expected utility representation. We then go on to discuss the necessary and sufficient condition for an individual's preferences to exhibit risk aversion under the expected utility representation assumption. Different measures of risk aversion will be proposed and used to analyze the comparative statics of an individual's portfolio behavior when faced with one risky asset and one riskless asset. Finally, we will discuss sufficient conditions for the comparative statics for the one risky and one riskless asset case to generalize to the case of multiple risky assets and a riskless asset.

Before we proceed, we note that throughout this book we will use positive, negative, greater than, smaller than, increasing, decreasing, and etc. to mean weak relations. When a relation is strict, we will emphasize it by using a "strictly" to modify it, for example, by using "strictly positive."

1.2 Suppose for now that there are two dates, time 0 and time 1, and there is a single consumption good available for consumption only at time 1. Uncertainty in the economy is modeled by uncertain states of nature to be realized at time 1. A state of nature is a complete description of the uncertain environment from time 0 to time 1. We denote the collection of all the possible states of nature by Ω and denote an element of Ω by ω. At time 0, individuals know that the true state of nature is an element of Ω but do not know which state will occur at time 1. A ***consumption plan*** is then a specification of the number of units of the single consumption good in different states of nature. Let x be a consumption plan. We will use x_ω to denote the number of units of the consumption good in state ω specified by x. When there are five states of nature denoted by $\omega_1, \ldots, \omega_5$. Table 1.2.1 tabulates a consumption plan x, which has 2 units of consumption in state ω_1, 3 units of consumption in states ω_2, etc. As we defined above, a consumption plan is a vector specifying units of consumption in different states of nature. Since the time 1 realized consumption is uncertain, a consumption plan x can also be viewed as a random variable and we will use $\tilde{x}$ to denote it when we use the "random variable" aspect of a consumption plan.

	ω_1	ω_2	ω_3	ω_4	ω_5
x_ω	2	3	1	8	0

Table 1.2.1: A Consumption Plan

An individual is represented by his preference relation $\succeq$ defined on a collection of consumption plans. We will formally define a preference relation shortly. Roughly, a preference relation is a mechanism that allows an individual to compare different consumption plans. For example, given two consumption plans, x and x', a preference relation enables an individual to tell whether he prefers x to x' or x' to x. For concreteness, we would like an individual's preferences to be represented by a ***utility function***, or H, in the sense that the individual prefers x to x' if and only if $H(x) \geq H(x')$. We will see later in this chapter that, under some regularity conditions, a preference relation can always be represented this way.

When the number of states is very large, a consumption plan x is a vector of large dimension and the function H will be complicated to analyze. It would be more convenient if there existed a function u that allowed comparison among consumption plans that are certain and a probability P that gave the relative likelihood of states of nature such that the preference relation can be represented as an ***expected utility*** in the sense that consumption plan x is preferred to consumption plan x' if and only if the expected utility of x is greater than the expected utility of x', that is,

$$\int_\Omega u(x_\omega)dP(\omega) \geq \int_\Omega u(x'_\omega)dP(\omega). \tag{1.2.1}$$

Denoting the expectation operator under P by $E[\cdot]$, (1.2.1) can be equivalently written as

$$E[u(\tilde{x})] \geq E[u(\tilde{x}')], \tag{1.2.2}$$

where we have used the random variable aspect of x and x'. Note that a consumption plan is ***certain*** if the number of units of consumption does not vary across different states of nature. Note also that in the above expected utility representation, if x and x' are both certain or ***sure things***, that is

$$x_\omega = z \quad \text{and } x'_\omega = z' \quad \forall \omega \in \Omega,$$

for some constants z and z', then $E[u(\tilde{x})] = u(z)$ and $E[u(\tilde{x}')] = u(z')$. In this sense, u compares consumption plans that are certain.

Certainly not all preference relations have an expected utility representation. Indeed, we have to put a fair amount of structure on a preference relation to achieve this purpose. In general, there are two approaches for a preference relation to have an expected utility representation, depending on whether one treats the probabilities of the states of nature as objective or subjective. The former approach was introduced by von Neumann and Morgenstern (1953) and the resulting function u is thus called the *von Neumann-Morgenstern utility function.* The latter approach was taken by Savage (1972), who views probability assessments as an integral part of an investor's preferences and thus purely subjective. However, the distinction between subjective and objective probability assessments is inconsequential to our purpose in this book. Hence for the analysis to follow, we will not distinguish between them and always call the function u defined on sure things a von Neumann–Morgenstern utility function.

Before discussing the representation of a preference relation by an expected utility, in the next section, we give a definition of a preference relation and discuss conditions under which a preference relation can be represented by a utility function H.

1.3. Formally, let X be the collection of consumption plans under consideration. A *binary relation* $\succeq$ on X is a collection of pairs of consumption plans (x, y). If (x, y) is in the relation, we write $x \succeq y$ and say x is preferred to y. If (x, y) is not in the relation, then we write $x \not\succeq y$ and say x is not preferred to y.

A binary relation is *transitive* if $x \succeq y$ and $y \succeq v$ imply $x \succeq v$; that is, if x is preferred to y and y is preferred to v, then x is preferred to v. A binary relation is said to be *complete* if for any two consumption plans x and y, we either have $x \succeq y$ or $y \succeq x$; that is, any two consumption plans can always be compared.

A *preference relation* is a binary relation that is transitive and complete. We can also define an indifference relation and a strict preference relation. Formally, given a preference relation $\succeq$, two consumption plans x and y are said to be *indifferent* to each other if $x \succeq y$ and $y \succeq x$, denoted by $x \sim y$. The consumption plan x

is said to be ***strictly preferred to y***, denoted by $x \succ y$, if $x \succeq y$ and $y \not\succeq x$. Note that a strict binary relation and an indifference relation can also be similarly defined for any given binary relation.

1.4. When X has a finite number of elements, a preference relation $\succeq$ can *always* be represented by a utility function. This assertion can be proved in a straightforward manner. The readers are asked to furnish a proof in Exercise 1.1. Here we shall give an example to demonstrate the essential idea.

Suppose that there are three consumption plans in X, denoted by x_1, x_2, and x_3. Pick any consumption plan, say x_3, and define

$$H(x_3) \equiv b,$$

where b is an arbitrary constant. Next take x_1. Since a preference relation is complete, x_1 and x_3 can be compared. We define

$$H(x_1) \equiv \begin{cases} b+1 & \text{if } x_1 \succ x_3; \\ b-1 & \text{if } x_3 \succ x_1; \\ b & \text{if } x_3 \sim x_1. \end{cases}$$

That is, compare x_1 and x_3. If x_1 is strictly preferred to x_3, we assign a value strictly larger than b to $H(x_1)$; and similarly for other cases. Without loss of generality, suppose that $x_1 \succ x_3$. Finally, we compare x_2 with x_1 and x_3, and define

$$H(x_2) \equiv \begin{cases} b-1 & \text{if } x_3 \succ x_2; \\ b & \text{if } x_2 \sim x_3; \\ b+\frac{1}{2} & \text{if } x_1 \succ x_2 \succ x_3; \\ b+1 & \text{if } x_2 \sim x_1\,; \\ b+2 & \text{if } x_2 \succ x_1\,. \end{cases}$$

Here we compare x_2 with x_1 and x_3 and assign values to $H(x_2)$ in a natural way. It should now be transparent that

$$H(x_n) \geq H(x_m) \quad \text{if and only if} \quad x_n \succeq x_m, \quad n, m = 1, 2, 3.$$

That is, H as defined above represents the preference relation $\succeq$.

When X has a countable number of elements, the above idea can be carried out in a similar way to conclude that a preference relation can always be represented by a utility function.

1.5. Matters are not as simple when an individual expresses his preferences on an uncountably infinite number of consumption plans. In such event, there exist well-known examples of preference relations that cannot be represented by utility functions. The so-called *Lexicographic preference relation* is one such example; we refer readers to Exercise 1.2 for a brief description. Thus, for general X, additional conditions on a preference relation will be needed for an expected utility representation to exist. It turns out that the additional condition needed is purely technical in nature and is stated in Exercise 1.2. Interested readers should consult Debreu (1954) and Fishburn (1970) for details.

1.6. Now we turn to the representation of a preference relation by an expected utility. Let P be a probability defined on the state space Ω, which can be either objective or subjective. (For the technically inclined readers, we are a bit informal here. When Ω has uncountably infinite elements, a probability is actually defined not on Ω but rather on a collection of subsets of Ω that satisfies a certain structure.) A consumption plan is a random variable, whose probabilistic characteristics are specified by P. We can define the distribution function for a consumption plan x as follows:

$$F_x(z) \equiv P\{\omega \in \Omega : x_\omega \le z\}.$$

If a preference relation $\succeq$ has an expected utility representation with a utility function u on sure things, the expected utility derived from x is

$$E[u(\tilde{x})] = \int_{-\infty}^{+\infty} u(z) dF_x(z).$$

From the above relation, we see that if two consumption plans x and x' have the same distribution function, they will yield the same expected utility and are indifferent to each other. This demonstrates that the primitive objects on which an individual expresses his or her preferences are probability distributions of consumption. Note that two consumption plans having the same distribution function can have very different consumption patterns across states of nature.

1.7. To simplify matters, we shall assume that an individual only expresses his preferences on probability distributions defined on a finite set Z. In other words, the collection of consumption plans X on which an individual expresses his or her preferences must have the property that

$$x_\omega \in Z \quad \forall \omega \in \Omega, \forall x \in X.$$

For example, if $Z = \{1, 2, 3\}$, then the units of consumption in any state can only be 1, 2, or 3. This assumption can be justified, for example, when the consumption commodity is not perfectly divisible and the supply of the commodity is finite. In this case, we can represent a consumption plan x by a function $p(\cdot)$ defined on Z, where $p(z)$ is the probability that x is equal to z. Thus $p(z) \geq 0$ for all $z \in Z$ and $\sum_{z \in Z} p(z) = 1$. The distribution function for the consumption plan x discussed in Section 1.6 is then

$$F_x(z') = \sum_{z \leq z'} p(z),$$

and

$$E[u(\tilde{x})] = \sum_{z \in Z} u(z)p(z).$$

One can also think of a consumption plan as a *lottery* with prizes in Z. The probability of getting a prize z is $p(z)$.

We denote the space of probabilities on Z by $\mathbf{P}$ and its elements by p, q, and r. If $p \in \mathbf{P}$, the probability of z under p is $p(z)$.

1.8. The following three behavioral axioms are necessary and sufficient for a binary relation defined on $\mathbf{P}$ to have an expected utility representation.

Axiom 1. $\succeq$ is a preference relation on $\mathbf{P}$.

Axiom 2. For all $p, q, r \in \mathbf{P}$ and $a \in (0, 1]$, $p \succ q$ implies $ap + (1 - a)r \succ aq + (1 - a)r$.