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Lectures on Discrete Geometry

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Preface

The next several pages describe the goals and the main topics of this book.

Questions in discrete geometry typically involve finite sets of points, lines, circles, planes, or other simple geometric objects. For example, one can ask, what is the largest number of regions into which n lines can partition the plane, or what is the minimum possible number of distinct distances occurring among n points in the plane? (The former question is easy, the latter one is hard.) More complicated objects are investigated, too, such as convex polytopes or finite families of convex sets. The emphasis is on "combinatorial" properties: Which of the given objects intersect, or how many points are needed to intersect all of them, and so on.

Many questions in discrete geometry are very natural and worth studying for their own sake. Some of them, such as the structure of 3-dimensional convex polytopes, go back to the antiquity, and many of them are motivated by other areas of mathematics. To a working mathematician or computer scientist, contemporary discrete geometry offers results and techniques of great diversity, a useful enhancement of the "bag of tricks" for attacking problems in her or his field. My experience in this respect comes mainly from combinatorics and the design of efficient algorithms, where, as time progresses, more and more of the first-rate results are proved by methods drawn from seemingly distant areas of mathematics and where geometric methods are among the most prominent.

The development of *computational geometry* and of geometric methods in *combinatorial optimization* in the last 20–30 years has stimulated research in discrete geometry a great deal and contributed new problems and motivation. Parts of discrete geometry are indispensable as a foundation for any serious study of these fields. I personally became involved in discrete geometry while working on geometric algorithms, and the present book gradually grew out of lecture notes initially focused on computational geometry. (In the meantime, several books on computational geometry have appeared, and so I decided to concentrate on the nonalgorithmic part.)

In order to explain the path chosen in this book for exploring its subject, let me compare discrete geometry to an Alpine mountain range. Mountains can be explored by bus tours, by walking, by serious climbing, by playing in the local casino, and in many other ways. The book should provide safe trails to a few peaks and lookout points (key results from various subfields of discrete geometry). To some of them, convenient paths have been marked in the literature, but for others, where only climbers' routes exist in research papers, I tried to add some handrails, steps, and ropes at the critical places, in the form of intuitive explanations, pictures, and concrete and elementary proofs.[1] However, I do not know how to build cable cars in this landscape: Reaching the higher peaks, the results traditionally considered difficult, still needs substantial effort. I wish everyone a clear view of the beautiful ideas in the area, and I hope that the trails of this book will help some readers climb yet unconquered summits by their own research. (Here the shortcomings of the Alpine analogy become clear: The range of discrete geometry is infinite and no doubt, many discoveries lie ahead, while the Alps are a small spot on the all too finite Earth.)

This book is primarily an *introductory textbook*. It does not require any special background besides the usual undergraduate mathematics (linear algebra, calculus, and a little of combinatorics, graph theory, and probability). It should be accessible to early graduate students, although mastering the more advanced proofs probably needs some mathematical maturity. The first and main part of each section is intended for teaching in class. I have actually taught most of the material, mainly in an advanced course in Prague whose contents varied over the years, and a large part has also been presented by students, based on my writing, in lectures at special seminars (Spring Schools of Combinatorics). A *short summary* at the end of the book can be useful for reviewing the covered material.

The book can also serve as a collection of *surveys* in several narrower subfields of discrete geometry, where, as far as I know, no adequate recent treatment is available. The sections are accompanied by remarks and bibliographic notes. For well-established material, such as convex polytopes, these parts usually refer to the original sources, point to modern treatments and surveys, and present a sample of key results in the area. For the less well covered topics, I have aimed at surveying most of the important recent results. For some of them, proof outlines are provided, which should convey the main ideas and make it easy to fill in the details from the original source.

Topics. The material in the book can be divided into several groups:

- *Foundations* (Sections 1.1–1.3, 2.1, 5.1–5.4, 5.7, 6.1). Here truly basic things are covered, suitable for any introductory course: linear and affine subspaces, fundamentals of convex sets, Minkowski's theorem on lattice points in convex bodies, duality, and the first steps in convex polytopes, Voronoi diagrams, and hyperplane arrangements. The remaining sections of Chapters 1, 2, and 5 go a little further in these topics.

[1] I also wanted to invent fitting names for the important theorems, in order to make them easier to remember. Only few of these names are in standard usage.

- *Combinatorial complexity of geometric configurations* (Chapters 4, 6, 7, and 11). The problems studied here include line–point incidences, complexity of arrangements and lower envelopes, Davenport–Schinzel sequences, and the k-set problem. Powerful methods, mainly probabilistic, developed in this area are explained step by step on concrete nontrivial examples. Many of the questions were motivated by the analysis of algorithms in computational geometry.
- *Intersection patterns and transversals of convex sets.* Chapters 8–10 contain, among others, a proof of the celebrated (p, q)-theorem of Alon and Kleitman, including all the tools used in it. This theorem gives a sufficient condition guaranteeing that all sets in a given family of convex sets can be intersected by a bounded (small) number of points. Such results can be seen as far-reaching generalizations of the well-known Helly's theorem. Some of the finest pieces of the weaponry of contemporary discrete and computational geometry, such as the theory of the VC-dimension or the regularity lemma, appear in these chapters.
- *Geometric Ramsey theory* (Chapters 3 and 9). Ramsey-type theorems guarantee the existence of a certain "regular" subconfiguration in every sufficiently large configuration; in our case we deal with geometric objects. One of the historically first results here is the theorem of Erdős and Szekeres on convex independent subsets in every sufficiently large point set.
- *Polyhedral combinatorics and high-dimensional convexity* (Chapters 12–14). Two famous results are proved as a sample of polyhedral combinatorics, one in graph theory (the weak perfect graph conjecture) and one in theoretical computer science (on sorting with partial information). Then the behavior of convex bodies in high dimensions is explored; the highlights include a theorem on the volume of an N-vertex convex polytope in the unit ball (related to algorithmic hardness of volume approximation), measure concentration on the sphere, and Dvoretzky's theorem on almost-spherical sections of convex bodies.
- *Representing finite metric spaces by coordinates* (Chapter 15). Given an n-point metric space, we would like to visualize it or at least make it computationally more tractable by placing the points in a Euclidean space, in such a way that the Euclidean distances approximate the given distances in the finite metric space. We investigate the necessary error of such approximation. Such results are of great interest in several areas; for example, recently they have been used in approximation algorithms in combinatorial optimization (multicommodity flows, VLSI layout, and others).

These topics surely do not cover all of discrete geometry, which is a rather vague term anyway. The selection is (necessarily) subjective, and naturally I preferred areas that I knew better and/or had been working in. (Unfortunately, I have had no access to supernatural opinions on proofs as a more reliable guide.) Many interesting topics are neglected completely, such as the wide area of packing and covering, where very accessible treatments exist, or the celebrated negative solution by Kahn and Kalai of the Borsuk conjecture, which I consider sufficiently popularized by now. Many more chapters analogous to the fifteen of this book could be added, and each of the fifteen chapters could be expanded into a thick volume. But the extent of the book, as well as the time for its writing, are limited.

Exercises. The sections are complemented by exercises. The little framed numbers indicate their difficulty: 1 is routine, 5 may need quite a bright idea. Some of the exercises used to be a part of homework assignments in my courses and the classification is based on some experience, but for others it is just an unreliable subjective guess. Some of the exercises, especially those conveying important results, are accompanied by hints given at the end of the book.

Additional results that did not fit into the main text are often included as exercises, which saves much space. However, this greatly enlarges the danger of making false claims, so the reader who wants to use such information may want to check it carefully.

Sources and further reading. A great inspiration for this book project and the source of much material was the book *Combinatorial Geometry* of Pach and Agarwal [PA95]. Too late did I become aware of the lecture notes by Ball [Bal97] on modern convex geometry; had I known these earlier I would probably have hesitated to write Chapters 13 and 14 on high-dimensional convexity, as I would not dare to compete with this masterpiece of mathematical exposition. Ziegler's book [Zie94] can be recommended for studying convex polytopes. Many other sources are mentioned in the notes in each chapter. For looking up information in discrete geometry, a good starting point can be one of the several handbooks pertaining to the area: *Handbook of Convex Geometry* [GW93], *Handbook of Discrete and Computational Geometry* [GO97], *Handbook of Computational Geometry* [SU00], and (to some extent) *Handbook of Combinatorics* [GGL95], with numerous valuable surveys. Many of the important new results in the field keep appearing in the journal *Discrete and Computational Geometry.*

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Errors. If you find errors in the book, especially serious ones, I would appreciate it if you would let me know (email: `matousek@kam.mff.cuni.cz`). I plan to post a list of errors at `http://www.ms.mff.cuni.cz/~matousek`.

Prague, July 2001 *Jiří Matoušek*

Notation and Terminology

This section summarizes rather standard things, and it is mainly for referen
More special notions are introduced gradually throughout the book. In or
to facilitate independent reading of various parts, some of the definitions
even repeated several times.

If X is a set, $|X|$ denotes the number of elements (cardinality) of X. If is a *multiset*, in which some elements may be repeated, then $|X|$ counts ea element with its multiplicity.

The very slowly growing function $\log^* x$ is defined by $\log^* x = 0$ for $x \le$ and $\log^* x = 1 + \log^*(\log_2 x)$ for $x > 1$.

For a real number x, $\lfloor x \rfloor$ denotes the largest integer less than or eq to x, and $\lceil x \rceil$ means the smallest integer greater than or equal to x. T boldface letters $\mathbf{R}$ and $\mathbf{Z}$ stand for the real numbers and for the intege respectively, while $\mathbf{R}^d$ denotes the d-dimensional Euclidean space. For a po $x = (x_1, x_2, \ldots, x_d) \in \mathbf{R}^d$, $\|x\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_d^2}$ is the Euclidean no of x, and for $x, y \in \mathbf{R}^d$, $\langle x, y \rangle = x_1y_1 + x_2y_2 + \cdots + x_dy_d$ is the scalar produ Points of $\mathbf{R}^d$ are usually considered as column vectors.

The symbol $B(x, r)$ denotes the closed ball of radius r centered at x some metric space (usually in $\mathbf{R}^d$ with the Euclidean distance), i.e., the of all points with distance at most r from x. We write B^n for the unit b $B(0, 1)$ in $\mathbf{R}^n$. The symbol ∂A denotes the boundary of a set $A \subseteq \mathbf{R}^d$, t is, the set of points at zero distance from both A and its complement.

For a measurable set $A \subseteq \mathbf{R}^d$, $\mathrm{vol}(A)$ is the d-dimensional Lebesgue m sure of A (in most cases the usual volume).

Let f and g be real functions (of one or several variables). The notat $f = O(g)$ means that there exists a number C such that $|f| \le C|g|$ for values of the variables. Normally, C should be an absolute constant, bu f and g depend on some parameter(s) that we explicitly declare to be fi (such as the space dimension d), then C may depend on these paramet as well. The notation $f = \Omega(g)$ is equivalent to $g = O(f)$, $f(n) = o(g($ to $\lim_{n\to\infty}(f(n)/g(n)) = 0$, and $f = \Theta(g)$ means that both $f = O(g)$ a $f = \Omega(g)$.

For a random variable X, the symbol $\mathbf{E}[X]$ denotes the expectation of and $\mathrm{Prob}[A]$ stands for the probability of an event A.

Graphs are considered simple and undirected in this book unless stated therwise, so a graph G is a pair (V, E), where V is a set (the *vertex set*) and $' \subseteq \binom{V}{2}$ is the *edge set*. Here $\binom{V}{k}$ denotes the set of all k-element subsets f V. For a *multigraph*, the edges form a multiset, so two vertices can be onnected by several edges. For a given (multi)graph G, we write $V(G)$ for e vertex set and $E(G)$ for the edge set. A *complete graph* has all possible dges; that is, it is of the form $\left(V, \binom{V}{2}\right)$. A complete graph on n vertices is enoted by K_n. A graph G is *bipartite* if the vertex set can be partitioned nto two subsets V_1 and V_2, the *(color) classes*, in such a way that each edge onnects a vertex of V_1 to a vertex of V_2. A graph $G' = (V', E')$ is a *subgraph* f a graph $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$. We also say that G *contains copy* of H if there is a subgraph G' of G isomorphic to H, where G' and I are *isomorphic* if there is a bijective map $\varphi\colon V(G') \to V(H)$ such that $u, v\} \in E(G')$ if and only if $\{\varphi(u), \varphi(v)\} \in E(H)$ for all $u, v \in V(G')$. The *egree* of a vertex v in a graph G is the number of edges of G containing v. n *r-regular graph* has all degrees equal to r. Paths and cycles are graphs as n the following picture,

nd a path or cycle in G is a subgraph isomorphic to a path or cycle, respecively. A graph G is *connected* if every two vertices can be connected by a ath in G.

We recall that a set $X \subseteq \mathbf{R}^d$ is *compact* if and only if it is closed and ounded, and that a continuous function $f\colon X \to \mathbf{R}$ defined on a compact X ttains its minimum (there exists $x_0 \in X$ with $f(x_0) \le f(x)$ for all $x \in X$).

The *Cauchy–Schwarz inequality* is perhaps best remembered in the form $x, y\rangle \le \|x\| \cdot \|y\|$ for all $x, y \in \mathbf{R}^n$.

A real function f defined on an interval $A \subseteq \mathbf{R}$ (or, more generally, on a onvex set $A \subseteq \mathbf{R}^d$) is *convex* if $f(tx + (1-t)y) \le tf(x) + (1-t)f(y)$ for all $, y \in A$ and $t \in [0, 1]$. Geometrically, the graph of f on $[x, y]$ lies below the egment connecting the points $(x, f(x))$ and $(y, f(y))$. If the second derivative atisfies $f''(x) \ge 0$ for all x in an (open) interval $A \subseteq \mathbf{R}$, then f is convex n A. *Jensen's inequality* is a straightforward generalization of the definition f convexity: $f(t_1x_1 + t_2x_2 + \cdots + t_nx_n) \le t_1f(x_1) + t_2f(x_2) + \cdots + t_nf(x_n)$ or all choices of nonnegative t_i summing to 1 and all $x_1, \ldots, x_n \in A$. Or in ntegral form, if μ is a probability measure on A and f is convex on A, we have $f\left(\int_A x \, d\mu(x)\right) \le \int_A f(x) \, d\mu(x)$. In the language of probability theory, if X s a real random variable and $f\colon \mathbf{R} \to \mathbf{R}$ is convex, then $f(\mathbf{E}[X]) \le \mathbf{E}[f(X)]$; or example, $(\mathbf{E}[X])^2 \le \mathbf{E}[X^2]$.

Contents

1

Convexity

We begin with a review of basic geometric notions such as hyperplanes and affine subspaces in $\mathbf{R}^d$, and we spend some time by discussing the notion of general position. Then we consider fundamental properties of convex sets in $\mathbf{R}^d$, such as a theorem about the separation of disjoint convex sets by a hyperplane and Helly's theorem.

1.1 Linear and Affine Subspaces, General Position

Linear subspaces. Let $\mathbf{R}^d$ denote the d-dimensional Euclidean space. The points are d-tuples of real numbers, $x = (x_1, x_2, \ldots, x_d)$.

The space $\mathbf{R}^d$ is a vector space, and so we may speak of linear subspaces, linear dependence of points, linear span of a set, and so on. A linear subspace of $\mathbf{R}^d$ is a subset closed under addition of vectors and under multiplication by real numbers. What is the geometric meaning? For instance, the linear subspaces of $\mathbf{R}^2$ are the origin itself, all lines passing through the origin, and the whole of $\mathbf{R}^2$. In $\mathbf{R}^3$, we have the origin, all lines and planes passing through the origin, and $\mathbf{R}^3$.

Affine notions. An arbitrary line in $\mathbf{R}^2$, say, is *not* a linear subspace unless it passes through 0. General lines are what are called *affine subspaces.* An affine subspace of $\mathbf{R}^d$ has the form $x + L$, where $x \in \mathbf{R}^d$ is some vector and L is a linear subspace of $\mathbf{R}^d$. Having defined affine subspaces, the other "affine" notions can be constructed by imitating the "linear" notions.

What is the *affine hull* of a set $X \subseteq \mathbf{R}^d$? It is the intersection of all affine subspaces of $\mathbf{R}^d$ containing X. As is well known, the linear span of a set X can be described as the set of all linear combinations of points of X. What is an *affine combination* of points $a_1, a_2, \ldots, a_n \in \mathbf{R}^d$ that would play an analogous role? To see this, we translate the whole set by $-a_n$, so that a_n becomes the origin, we make a linear combination, and we translate back by

$+a_n$. This yields an expression of the form $\beta_1(a_1 - a_n) + \beta_2(a_2 - a_n) + \cdots + \beta_n(a_n - a_n) + a_n = \beta_1 a_1 + \beta_2 a_2 + \cdots + \beta_{n-1}a_{n-1} + (1 - \beta_1 - \beta_2 - \cdots - \beta_{n-1})a_n$, where $\beta_1, \ldots, \beta_n$ are arbitrary real numbers. Thus, an affine combination of points $a_1, \ldots, a_n \in \mathbf{R}^d$ is an expression of the form

$$\alpha_1 a_1 + \cdots + \alpha_n a_n, \text{ where } \alpha_1, \ldots, \alpha_n \in \mathbf{R} \text{ and } \alpha_1 + \cdots + \alpha_n = 1.$$

Then indeed, it is not hard to check that the affine hull of X is the set of all affine combinations of points of X.

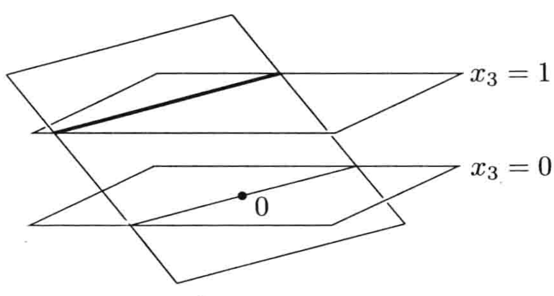
The *affine dependence* of points $a_1, \ldots, a_n$ means that one of them can be written as an affine combination of the others. This is the same as the existence of real numbers $\alpha_1, \alpha_2, \ldots \alpha_n$, at least one of them nonzero, such that both

$$\alpha_1 a_1 + \alpha_2 a_2 + \cdots + \alpha_n a_n = 0 \text{ and } \alpha_1 + \alpha_2 + \cdots + \alpha_n = 0.$$

(Note the difference: In an affine *combination*, the α_i sum to 1, while in an affine *dependence*, they sum to 0.)

Affine dependence of $a_1, \ldots, a_n$ is equivalent to linear dependence of the $n-1$ vectors $a_1 - a_n, a_2 - a_n, \ldots, a_{n-1} - a_n$. Therefore, the maximum possible number of affinely independent points in $\mathbf{R}^d$ is $d+1$.

Another way of expressing affine dependence uses "lifting" one dimension higher. Let $b_i = (a_i, 1)$ be the vector in $\mathbf{R}^{d+1}$ obtained by appending a new coordinate equal to 1 to a_i; then $a_1, \ldots, a_n$ are affinely dependent if and only if $b_1, \ldots, b_n$ are linearly dependent. This correspondence of affine notions in $\mathbf{R}^d$ with linear notions in $\mathbf{R}^{d+1}$ is quite general. For example, if we identify $\mathbf{R}^2$ with the plane $x_3 = 1$ in $\mathbf{R}^3$ as in the picture,



then we obtain a bijective correspondence of the k-dimensional linear subspaces of $\mathbf{R}^3$ that do not lie in the plane $x_3 = 0$ with $(k-1)$-dimensional affine subspaces of $\mathbf{R}^2$. The drawing shows a 2-dimensional linear subspace of $\mathbf{R}^3$ and the corresponding line in the plane $x_3 = 1$. (The same works for affine subspaces of $\mathbf{R}^d$ and linear subspaces of $\mathbf{R}^{d+1}$ not contained in the subspace $x_{d+1} = 0$.)

This correspondence also leads directly to extending the affine plane $\mathbf{R}^2$ into the *projective plane*: To the points of $\mathbf{R}^2$ corresponding to nonhorizontal lines through 0 in $\mathbf{R}^3$ we add points "at infinity," that correspond to horizontal lines through 0 in $\mathbf{R}^3$. But in this book we remain in the affine space most of the time, and we do not use the projective notions.

Let $a_1, a_2, \ldots, a_{d+1}$ be points in $\mathbf{R}^d$, and let A be the $d \times d$ matrix with $a_i - a_{d+1}$ as the ith column, $i = 1, 2, \ldots, d$. Then $a_1, \ldots, a_{d+1}$ are affinely independent if and only if A has d linearly independent columns, and this is equivalent to $\det(A) \neq 0$. We have a useful criterion of affine independence using a determinant.

Affine subspaces of $\mathbf{R}^d$ of certain dimensions have special names. A $(d-1)$-dimensional affine subspace of $\mathbf{R}^d$ is called a *hyperplane* (while the word *plane* usually means a 2-dimensional subspace of $\mathbf{R}^d$ for any d). One-dimensional subspaces are lines, and a k-dimensional affine subspace is often called a k-*flat.*

A hyperplane is usually specified by a single linear equation of the form $a_1x_1 + a_2x_2 + \cdots + a_dx_d = b$. We usually write the left-hand side as the scalar product $\langle a, x \rangle$. So a hyperplane can be expressed as the set $\{x \in \mathbf{R}^d\colon \langle a, x \rangle = b\}$ where $a \in \mathbf{R}^d \setminus \{0\}$ and $b \in \mathbf{R}$. A (closed) *half-space* in $\mathbf{R}^d$ is a set of the form $\{x \in \mathbf{R}^d\colon \langle a, x \rangle \geq b\}$ for some $a \in \mathbf{R}^d \setminus \{0\}$; the hyperplane $\{x \in \mathbf{R}^d\colon \langle a, x \rangle = b\}$ is its boundary.

General k-flats can be given either as intersections of hyperplanes or as affine images of $\mathbf{R}^k$ (parametric expression). In the first case, an intersection of k hyperplanes can also be viewed as a solution to a system $Ax = b$ of linear equations, where $x \in \mathbf{R}^d$ is regarded as a column vector, A is a $k \times d$ matrix, and $b \in \mathbf{R}^k$. (As a rule, in formulas involving matrices, we interpret points of $\mathbf{R}^d$ as *column* vectors.)

An *affine mapping* $f\colon \mathbf{R}^k \to \mathbf{R}^d$ has the form $f\colon y \mapsto By + c$ for some $d \times k$ matrix B and some $c \in \mathbf{R}^d$, so it is a composition of a linear map with a translation. The image of f is a k'-flat for some $k' \leq \min(k, d)$. This k' equals the rank of the matrix B.

General position. *"We assume that the points (lines, hyperplanes,...) are in general position."* This magical phrase appears in many proofs. Intuitively, general position means that no "unlikely coincidences" happen in the considered configuration. For example, if 3 points are chosen in the plane without any special intention, "randomly," they are unlikely to lie on a common line. For a planar point set in general position, we always require that no three of its points be collinear. For points in $\mathbf{R}^d$ in general position, we assume similarly that no unnecessary affine dependencies exist: No $k \leq d+1$ points lie in a common $(k-2)$-flat. For lines in the plane in general position, we postulate that no 3 lines have a common point and no 2 are parallel.

The precise meaning of general position is not fully standard: It may depend on the particular context, and to the usual conditions mentioned above we sometimes add others where convenient. For example, for a planar point set in general position we can also suppose that no two points have the same x-coordinate.

What conditions are suitable for including into a "general position" assumption? In other words, what can be considered as an unlikely coincidence? For example, let X be an n-point set in the plane, and let the coordinates of the ith point be (x_i, y_i). Then the vector $v(X) = (x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n)$ can be regarded as a point of $\mathbf{R}^{2n}$. For a configuration X in which $x_1 = x_2$, i.e., the first and second points have the same x-coordinate, the point $v(X)$ lies on the hyperplane $\{x_1 = x_2\}$ in $\mathbf{R}^{2n}$. The configurations X where *some* two points share the x-coordinate thus correspond to the union of $\binom{n}{2}$ hyperplanes in $\mathbf{R}^{2n}$. Since a hyperplane in $\mathbf{R}^{2n}$ has ($2n$-dimensional) measure zero, almost all points of $\mathbf{R}^{2n}$ correspond to planar configurations X with all the points having distinct x-coordinates. In particular, if X is any n-point planar configuration and $\varepsilon > 0$ is any given real number, then there is a configuration X', obtained from X by moving each point by distance at most ε, such that all points of X' have distinct x-coordinates. Not only that: Almost all small movements (*perturbations*) of X result in X' with this property.

This is the key property of general position: Configurations in general position lie arbitrarily close to any given configuration (and they abound in any small neighborhood of any given configuration). Here is a fairly general type of condition with this property. Suppose that a configuration X is specified by a vector $t = (t_1, t_2, \ldots, t_m)$ of m real numbers (coordinates). The objects of X can be points in $\mathbf{R}^d$, in which case $m = dn$ and the t_j are the coordinates of the points, but they can also be circles in the plane, with $m = 3n$ and the t_j expressing the center and the radius of each circle, and so on. The general position condition we can put on the configuration X is $p(t) = p(t_1, t_2, \ldots, t_m) \neq 0$, where p is some nonzero polynomial in m variables. Here we use the following well-known fact (a consequence of Sard's theorem; see, e.g., Bredon [Bre93], Appendix C): *For any nonzero m-variate polynomial $p(t_1, \ldots, t_m)$, the zero set $\{t \in \mathbf{R}^m \colon p(t) = 0\}$ has measure 0 in $\mathbf{R}^m$.*

Therefore, almost all configurations X satisfy $p(t) \neq 0$. So any condition that can be expressed as $p(t) \neq 0$ for a certain polynomial p in m real variables, or, more generally, as $p_1(t) \neq 0$ or $p_2(t) \neq 0$ or $\ldots$, for finitely or countably many polynomials $p_1, p_2, \ldots$, can be included in a general position assumption.

For example, let X be an n-point set in $\mathbf{R}^d$, and let us consider the condition "no $d+1$ points of X lie in a common hyperplane." In other words, no $d+1$ points should be affinely dependent. As we know, the affine dependence of $d+1$ points means that a suitable $d \times d$ determinant equals 0. This determinant is a polynomial (of degree d) in the coordinates of these $d+1$ points. Introducing one polynomial for every $(d+1)$-tuple of the points, we obtain $\binom{n}{d+1}$ polynomials such that at least one of them is 0 for any configuration X with $d+1$ points in a common hyperplane. Other usual conditions for general position can be expressed similarly.