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STOCHASTIC EQUATIONS THROUGH THE EYE OF THE PHYSICIST

Basic Concepts, Exact Results and Asymptotic Approximations

V. I. KLYATSKIN

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STOCHASTIC EQUATIONS THROUGH THE EYE OF THE PHYSICIST

Basic Concepts, Exact Results and Asymptotic Approximations

Translated from Russian
by A. Vinogradov

To Sonya Klyatskina

Preface

The book gives the theory of stochastic equations (including ordinary differential equations, partial differential equations, boundary-value problems, and integral equations) in terms of the functional analysis. The developed approach yields exact solutions to stochastic problems for a number of models of fluctuating parameters among which are telegrapher's and generalized telegrapher's processes, Markovian processes with a finite number of states, Gaussian Markovian processes, and functions of the above processes. Asymptotic methods of analyzing stochastic dynamic systems, such as delta-correlated random process (field) approximation and diffusion approximation are also considered. These methods are used to describe the coherent phenomena in stochastic systems (particle and passive tracer clustering in random velocity field, dynamic localization of plane waves in randomly layered media, and caustic structure formation in multidimensional random media).

The book is destined for scientists dealing with stochastic dynamic systems in different areas, such as hydrodynamics, acoustics, radio wave physics, theoretical and mathematical physics, and applied mathematics, and can be useful for senior and postgraduate students.

Now, a few words are due on the structure of the text. The book is in five parts.

The first part may be viewed as an introductory text. It takes up a few typical physical problems to discuss their solutions obtained under random perturbations of parameters affecting the system behavior. More detailed formulations of these problems and relevant statistical analysis may be found in other parts of the book.

The second part is devoted to the general theory of statistical analysis of dynamic systems with fluctuating parameters described by differential and integral equations. This theory is illustrated by analyzing specific dynamic systems.

The third part treats asymptotic methods of statistical analysis such as the delta-correlated random process (field) approximation and diffusion approximation.

The fourth part deals with analysis of specific physical problems associated with coherent phenomena. These are clustering and diffusion of particles and passive ingredients in a random velocity field, dynamic localization of plane waves propagating in layered random media, and formation of caustics by waves propagating in random multidimensional media. These phenomena are described by ordinary differential equations and partial differential equations. Each of these formulations splits into many separate problems of individual physical interest.

In order to avoid crowding the book by mathematical niceties, it is appended by the fifth part that consists of three appendixes presenting detailed derivations of some mathematical expressions used in the text. Specifically, they give a definition and some rules to calculate variational derivatives; they discuss the properties of wavefield factorization in a homogeneous space and in layered media which drastically simplify analysis of statistical problems. In these appendixes, we also discuss a derivation of the method of imbedding that offers a possibility of reformulating boundary-value wave problems into initial value problems with respect to auxiliary variables.

It is worth noting that purely mathematical and physical papers devoted to considered issues run into thousands. It would be physically impossible to give an exhaustive bibliography. Therefore, in this book we confine ourselves to referencing those papers which are used or discussed in this book and also recent review papers and with extensive bibliography on the subject.

V. I. Klyatskin

Moscow

Mephistopheles (about algebra)

. . .
I want to warn you:
    careful you should
Be with this science
    which is very tricky.
It can involve your brain into the chaos
    of vain unnecessary transformations.
If you don't manage
    to embrace its basis,
You won't be able indexes to differ.
. . .
What's important -
    deepen into symbols
And, having mastered them,
    you bravely step
Onto the path, which leads into the Kingdom
of Formulas, eternal and exact.

Kurd Lasswitz, Prost!,
www. gutenberg2000.de/lasswitz/prost/prost.htm
Translated by Alla Parollo.

Introduction

Different areas of physics pose statistical problems in ever-greater numbers. Apart from issues traditionally obtained in statistical physics, many applications call for including fluctuation effects into consideration. While fluctuations may stem from different sources (such as thermal noise, instability, and turbulence), methods used to treat them are very similar. In many cases, the statistical nature of fluctuations may be deemed known (either from physical considerations or from problem formulation) and the physical processes may be modeled by differential, integro-differential or integral equations.

Today the most powerful tools used to tackle complicated statistical problems are the *Markov theory of random processes* and the theory of *diffusion type processes* evolved from *Brownian motion theory*. Mathematical aspects underlying these theories and their applications have been treated extensively in academic literature and textbooks ([63]), and therefore we will not dwell on these issues in this treatise.

We will consider a statistical theory of dynamic and wave systems with fluctuating parameters. These systems can be described by ordinary differential equations, partial differential equations, integro-differential equations and integral equations. A popular way to solve such systems is by obtaining a closed system of equations for statistical characteristics of such systems to study their solutions as comprehensively as possible.

We note that often wave problems are boundary-value problems. When this is the case, one may resort to the imbedding method to reformulate the equations at hand to initial-value problems, thus considerably simplifying the statistical analysis [136].

We shall dwell in depth on dynamic systems whose fluctuating parameters are Gaussian random processes (fields), although what we present in this book is a general theory valid for fluctuating parameters of any nature.

The purpose of this book is to demonstrate how different physical problems described by stochastic equations may be solved on the base of a general approach. This treatment reveals interesting similarities between different physical problems.

Examples of specific physical systems outlined below are mainly borrowed from statistical hydrodynamics, statistical radio wave physics and acoustics because of author's research in these fields. However, similar problems and solution techniques occur in such areas as plasma physics, solid-state physics, magnetofluid dynamics to name a few.

In stochastic problems with fluctuating parameters, the variables are functions. It would be natural therefore to resort to functional methods for their analysis. We will use a *functional method* devised by Novikov [255] for Gaussian fluctuations of parameters in a turbulence theory and developed by the author of this book [132], [134]–[136] for the general case of dynamic systems and fluctuating parameters of arbitrary nature.

However, only a few dynamic systems lend themselves to analysis yielding solutions in a general form. It proved to be more efficient to use an asymptotic method where the statistical characteristics of dynamic problem solutions are expanded in powers of a small parameter which is essentially a ratio of the random impact's correlation time to the time of observation or to other characteristic time scale of the problem (in some cases, these may be spatial rather than temporal scales). This method is essentially a generalization of the theory of Brownian motion. It is termed the *delta-correlated random process (field) approximation.* In Brownian motion theory, this approximation is consistent with a model obtained by neglecting the time between random collisions as compared to all other time scales.

For dynamic systems described by ordinary differential stochastic equations with Gaussian fluctuations of parameters, this method leads to a Markovian problem solving model, and the respective equation for transition probability density has the form of the *Fokker–Planck equation.* In this book, we will consider in depth the methods of analysis available for this equation and its boundary conditions. We will analyze solutions and validity conditions by way of integral transformations. In more complicated problems described by partial differential equations, this method leads to a generalized equation of Fokker–Planck type in which variables are the derivatives of the solution's characteristic functional. For dynamic problems with non-Gaussian fluctuations of parameters, this method also yields Markovian type solutions. Under the circumstances, the probability density of respective dynamic stochastic equations satisfies a closed operator equation. For example, systems with parameters fluctuating in a Poisson profile are converted into the *Kolmogorov–Feller* type of integro-differential equations.

In physical investigations, Fokker–Planck and similar equations are usually set up from rule of thumb considerations, and dynamic equations are invoked only to calculate the coefficients of these equations. This approach is inconsistent, generally speaking. Indeed, the statistical problem is completely defined by dynamic equations and assumptions on the statistics of random impacts. For example, the Fokker–Planck equation must be a logical sequence of the dynamic equations and some assumptions on the character of random impacts. It is clear that not all problems lend themselves for reducing to a Fokker–Planck equation. The functional approach allows one to derive a Fokker–Planck equation from the problem's dynamic equation along with its applicability conditions.

For a certain class of random processes (Markovian telegrapher's processes, Gaussian Markovian process and the like), the developed functional approach also yields closed equations for the solution probability density with allowance for a finite correlation time of random interactions.

For processes with Gaussian fluctuations of parameters, one may construct a better physical approximation than the delta-correlated random process (field) approximation, — the *diffusion approximation* that allows for finiteness of correlation time radius. In this approximation, the solution is Markovian and its applicability condition has transparent physical meaning, namely, the statistical effects should be small within the correlation time of fluctuating parameters. This book treats these issues in depth from a general standpoint and for some specific physical applications.

In recent time, the interest of both theoreticians and experimenters has been attracted to relation of the behavior of average statistical characteristics of a problem solution with the behavior of the solution in certain happenings (realizations). This is especially important for geophysical problems related to the atmosphere and ocean where, generally speaking, a respective averaging ensemble is absent and experimenters, as a rule, have to do with individual observations.

Seeking solutions to dynamic problems for these specific realizations of medium parameters is almost hopeless due to extreme mathematical complexity of these problems.

At the same time, researchers are interested in main characteristics of these phenomena without much need to know specific details. Therefore, the idea to use a well developed approach to random processes and fields based on ensemble averages rather than separate observations proved to be very fruitful. By way of example, almost all physical problems of atmosphere and ocean to some extent are treated by statistical analysis.

Randomness in medium parameters gives rise to a stochastic behavior of physical fields. Individual samples of scalar two-dimensional fields $\rho(\mathbf{R}, t)$, $\mathbf{R} = (x, y)$, say, recall a rough mountainous terrain with randomly scattered peaks, troughs, ridges and saddles. Common methods of statistical averaging (computing mean-type averages — $\langle \rho(\mathbf{R}, t) \rangle$, space-time correlation function — $\langle \rho(\mathbf{R}, t)\, \rho(\mathbf{R}', t') \rangle$ etc., where $\langle ... \rangle$ implies averaging over an ensemble of random parameter samples) smooth the qualitative features of specific samples. Frequently, these statistical characteristics have nothing in common with the behavior of specific samples, and at first glance may even seem to be at variance with them. For example, the statistical averaging over all observations makes the field of average concentration of a passive tracer in a random velocity field ever more smooth, whereas each its realization sample tends to be more irregular in space due to mixture of areas with substantially different concentrations.

Thus, these types of statistical average usually characterize 'global' space-time dimensions of the area with stochastic processes but tell no details about the process behavior inside the area. For this case, details heavily depend on the velocity field pattern, specifically, on whether it is divergent or solenoidal. Thus, the first case will show with the total probability that *clusters* will be formed, i.e. compact areas of enhanced concentration of tracer surrounded by vast areas of low-concentration tracer. In the circumstances, all statistical moments of the distance between the particles will grow with time exponentially; that is, on average, a statistical recession of particles will take place.

In a similar way, in case of waves propagating in random media, an exponential spread of the rays will take place on average; but simultaneously, with the total probability, *caustics* will form at finite distances. One more example to illustrate this point is the *dynamic localization* of plane waves in layered randomly inhomogeneous media. In this phenomenon, the wavefield intensity exponentially decays inward the medium with the probability equal to unity when the wave is incident on the half-space of such a medium, while all statistical moments increase exponentially with distance from the boundary of the medium.

These physical processes and phenomena occurring with the probability equal to unity will be referred to as *coherent* processes and phenomena [157]. This type of *statistical coherence* may be viewed as some organization of the complex dynamic system, and retrieval of its *statistically stable characteristics* is similar to the concept of *coherence* as *self-organization* of multicomponent systems that evolve from the random interactions of their elements [254]. In the general case, it is rather difficult to say whether or not the phenomenon occurs with the probability equal to unity. However, for a number of applications amenable to treatment with the simple models of fluctuating parameters, this may be handled by analytical means. In other cases, one may verify this by performing numerical modeling experiments or analyzing experimental findings.

The complete statistic (say, the whole body of all n-point space-time moment functions), would undoubtedly contain all the information about the investigated dynamic system. In practice, however, one may succeed only in studying the simplest statistical characteristics associated mainly with one-time and one-point probability distributions. It would be reasonable to ask how with these statistics on hand one would look into the quantitative and qualitative behavior of some system happenings?

This question is answered by *methods of statistical topography.* These methods were highlighted by [319], who seems to had coined this term. Statistical topography yields a different philosophy of statistical analysis of dynamic stochastic systems, which may prove useful for experimenters planning a statistical processing of experimental data. These issues are treated in depths in this book.

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