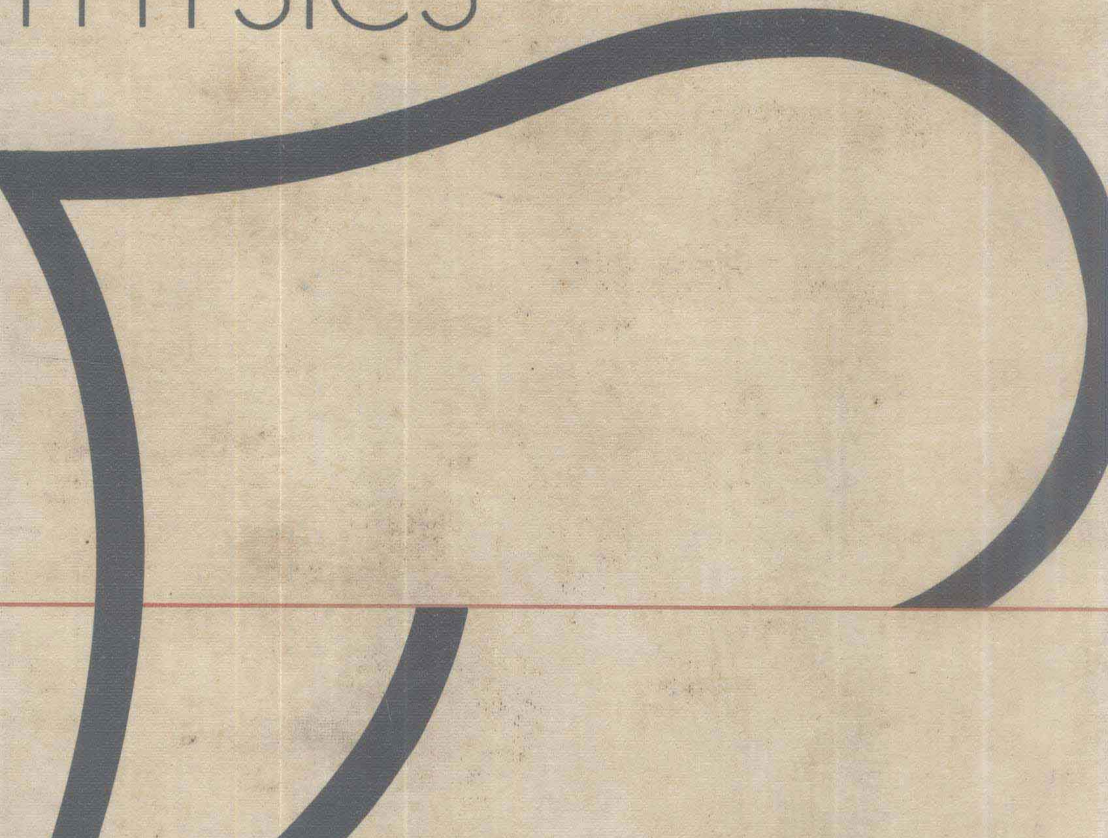
LINEAR EQUATIONS OF MATHEMATICAL PHYSICS



MIKHLIN

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PREFACE

This book is designed as a text and reference for mathematicians, physicists, and engineers who need to solve problems of mathematical physics or to use the mathematical principles involved in their applied or basic research work.

The linear differential equations of mathematical physics constitute one of the most extensive branches of analysis. A large number of monographs and textbooks as well as almost innumerable periodical articles have been devoted to it. At the same time, this is a branch of analysis with many ramifications interwoven with other branches of this subject and with mathematics in general. In the arsenal of present-day mathematical tools, we have topology and special functions, functional analysis and the classical theory of functions of a complex variable, theory of functions of a real variable, and approximation techniques. Mathematical physics is closely connected both with the most abstract divisions of contemporary mathematics and with the most concrete applications to the problems of physics and technology.

Naturally, such a complex subject has resulted in different methods of exposition. Together with the simple material contained in Chapters 1 to 4, less elementary material is given in Chapters 5 and 9.

The present text follows more or less the usual construction of courses in mathematical physics. Chapter 1 gives the overall properties of the equations of mathematical physics, the reduction of the equations to canonical form, and the classification of these equations. The Cauchy problem is stated and some general remarks concerning the statement of other boundary-value problems are made.

Chapter 2 discusses the basic problems leading to hyperbolic equations and the solution of the basic problems for hyperbolic equations with two independent variables. The problems associated with the solution of the wave equation are treated in detail. The Fourier method is explained in detail for equations with two or more independent variables. The most important facts concerning hyperbolic equations and systems of equations of a more general form are presented.

Chapters 3 and 4 present the elementary facts concerning the equations of Laplace, Poisson, and Helmholtz and their solutions. A great deal of attention is given to the results obtained by the method of separation of variables. Chapter 5 is devoted to more general equations (and systems of equations) of the elliptic type. Elliptic equations with a small parameter serving as coefficient of the highest derivatives are also studied in Chapter 5.

Chapter 6 takes up parabolic equations and systems of equations. The first five sections of this chapter deal with elementary information concerning the heat-flow equation. The following sections of this chapter are devoted to the general theory and are less elementary.

The remaining chapters of the book take the reader somewhat beyond the usual courses in mathematical physics. Chapters 7 and 8 have similar subjects: Chapter 7 deals with degenerate hyperbolic and elliptic equations and Chapter 8 has to do with equations of mixed elliptic-hyperbolic type. Equations of this type play a significant part in gas-dynamics, for example.

Chapter 9 is devoted to diffraction theory, which is of interest in connection with numerous applications. The problem is studied in detail for the wave equation, Maxwell's equations, and the dynamic equations of elasticity theory. Certain less simple questions dealing with the theory of propagation of waves are examined in this chapter.

The present book does not deal with the studies made in recent years on the most general systems of partial differential equations. It seemed to us that the theory of such systems is not yet sufficiently complete. We also have omitted the numerous and important investigations on the spectra of elliptic differential operators, such as, for example, the Schroedinger operator.

S. G. Mikhlin
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June 1966

TABLE OF CONTENTS

LIST OF SYMBOLS

CHAPTER I

The General Properties of Linear Partial Differential Equations

1. Basic Concepts and Definitions

A differential equation that contains, in addition to the independent variables and the unknown function, one or more partial derivatives of the unknown function is called a **partial differential equation.**

The highest order of any of the partial derivatives in the equation is the order of the differential equation.

From a standpoint of mathematical physics, the most important and most thoroughly studied equations are those of second order. In the case of two independent variables, a second-order equation can be written in the following general form:

$$F\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y^2}, \frac{\partial^2 u}{\partial x\, \partial y}\right) = 0 \tag{1.1}$$

An equation is said to be **linear** if it is linear in the unknown function and all its derivatives. A linear second-order equation with two independent variables has the following general form:

$$A(x, y)\frac{\partial^2 u}{\partial x^2} + 2B(x, y)\frac{\partial^2 u}{\partial x\, \partial y} + C(x, y)\frac{\partial^2 u}{\partial y^2} + a(x, y)\frac{\partial u}{\partial x} + b(x, y)\frac{\partial u}{\partial y} + c(x, y)u = f(x, y) \tag{1.2}$$

where $A(x, y)$, $B(x, y)$, $\cdots$, $c(x, y)$, and $f(x, y)$ are given functions of the variables x and y.

If $f(x, y) \equiv 0$, Equation (1.2) is said to be **homogeneous**; otherwise, it is **nonhomogeneous.**

Let us consider an equation that is linear in the derivatives of highest order. In the case of two independent variables, such an equation is of the form

$$A(x, y)\frac{\partial^2 u}{\partial x^2} + 2B(x, y)\frac{\partial^2 u}{\partial x\, \partial y} + C(x, y)\frac{\partial^2 u}{\partial y^2} = \Phi\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) \tag{1.3}$$

If the coefficients A, B, and C depend not only on x and y but also on u, $\partial u/\partial x$, and $\partial u/\partial y$, the equation is said to be **quasilinear**. A linear equation is a special case of a quasilinear equation.

A second-order linear equation with n independent variables can be written in the following general form:

$$\sum_{i,j=1}^{n} A_{ij}\frac{\partial^2 u}{\partial x_i\,\partial x_j}+\sum_{i=1}^{n} B_i\frac{\partial u}{\partial x_i}+Cu=f \qquad A_{ij}=A_{ji} \tag{1.4}$$

where the A_{ij}, B_i, C, and f are given functions of the n independent variables.

Let us consider a partial differential equation of order m. A function u defined in some region D in the space of the independent variables is called a **solution** or an **integral** of the given equation in the region D if throughout this region the function u has continuous partial derivatives up to order m inclusively and substitution of u into the original equation reduces that equation to an identity.

The requirement that the first m partial derivatives exist is often unjustified from a physical, and sometimes even from a mathematical, point of view. Therefore, in addition to the concept of a "classical" solution that we have just presented, the concept of a generalized solution of a partial differential equation has been introduced.

We now give the simplest definition of a generalized solution. If there exists a sequence of classical solutions of the given differential equation in D and if this sequence converges uniformly to some function u in an arbitrary subregion in the interior of the region D, this function u is said to be a **generalized solution** of the given differential equation in the region D. The concept of a generalized solution was introduced by S. L. Sobolev.

Consider a linear equation of order m in n independent variables, which we denote by $x_1, x_2, \cdots, x_n$. As before, we denote the unknown function by u. By transposing the terms not containing the unknown to the right side of the equation and leaving those terms with the unknown function and its derivatives on the left, we reduce the given equation to the form

$$L[u]=f(x_1, x_2, \cdots, x_n) \tag{1.5}$$

If $f\equiv 0$, the equation is homogeneous and is of the form

$$L[u]=0 \tag{1.6}$$

Sometimes, the square brackets are dropped and one writes simply Lu.

The symbol $L[u]$ is called a **linear differential operator** operating on the function u.

Linear differential operators have the following two properties*:

* *These properties in fact* define *the concept of linearity of operators.*

(1) A constant factor may be taken outside the symbol for the operator:

$$L[cu] = cL[u] \tag{1.7}$$

(2) Application of the operator to the sum of two functions yields the sum of the results obtained by applying the operator to the individual functions:

$$L[u_1 + u_2] = L[u_1] + L[u_2] \tag{1.8}$$

For *homogeneous* equations:

(1) If u is a solution and C is a constant, the product Cu is also a solution.

(2) If u_1 and u_2 are solutions, the sum $u_1 + u_2$ is also a solution.

Property (2) can be extended to the sum of an arbitrary finite number of terms. If we have an infinite sequence of solutions (u_n), the series

$$\sum_{n=1}^{\infty} u_n$$

is called a **formal solution** whether or not it converges. If the solutions u_n are classical solutions, if the series converges uniformly, and if the function representing its sum has partial derivatives of the necessary orders, this sum will be a classical solution of Equation (1.6). If the series converges uniformly but its sum does not have the necessary partial derivatives, this sum will be a generalized solution of Equation (1.6).

If a classical solution $u(x, \alpha)$ is an integrable function of a parameter α, the integral

$$\int C(\alpha)u(x, \alpha)\, d\alpha$$

where $C(\alpha)$ is an arbitrary continuous function of α and the limits of integration are independent of x, may be either a classical solution (if the integral converges uniformly and has the necessary partial derivatives) or a generalized solution (if the integral converges uniformly but does not have the necessary derivatives).

For *nonhomogeneous* equations:

(1) If u is a solution of a nonhomogeneous equation and v is a solution of the corresponding homogeneous equation, the sum $u + v$ is a solution of the nonhomogeneous equation.

(2) If u_1 is a solution of a nonhomogeneous equation whose right member is f_1 and if u_2 is a solution of a nonhomogeneous equation (with the same left member as the preceding equation) whose right member is f_2, then $u_1 + u_2$ is a solution of the equation (still with the same left member) whose right member is $f_1 + f_2$. This property can be extended to the sum of an arbitrary finite number of terms.

Consider the linear second-order differential operator

$$L[u] \equiv \sum_{i,j=1}^{n} A_{ij} \frac{\partial^2 u}{\partial x_i \, \partial x_j} + \sum_{i=1}^{n} B_i \frac{\partial u}{\partial x_i} + Cu \tag{1.9}$$

Suppose that the coefficient C is continuous in some region D, that the coefficients B_i are continuously differentiable, and that the coefficients A_{ij} and the function u are twice continuously differentiable. The operator

$$M[v] \equiv \sum_{i,j=1}^{n} \frac{\partial^2 (A_{ij} v)}{\partial x_i \, \partial x_j} - \sum_{i=1}^{n} \frac{\partial (B_i v)}{\partial x_i} + Cv \tag{1.10}$$

is said to be the **adjoint** of the operator L. The original operator L is the adjoint of the operator M.

If the operator L coincides with M, it is said to be **self-adjoint.**

A self-adjoint operator may be reduced to the form

$$L[u] = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(A_{ij} \frac{\partial u}{\partial x_j} \right) + Cu$$

We see that the following expression reduces to a divergence

$$vL[u] - uM[v] = \sum_{i=1}^{n} \frac{\partial P_i}{\partial x_i} \tag{1.11}$$

where $$P_i = \sum_{j=1}^{n} \left(vA_{ij} \frac{\partial u}{\partial x_j} - u \frac{\partial (A_{ij} v)}{\partial x_j} \right) + B_i uv \qquad i = 1, 2, \cdots, n \tag{1.12}$$

If D is a finite region bounded by a sufficiently regular closed surface S, we have by the divergence theorem

$$\int_D (vL[u] - uM[v]) \, dx = \int_S \sum_{i=1}^{n} P_i \cos(\nu, x_i) \, dS \tag{1.13}$$

Here, we use the following notations: dx denotes an element of volume in the coordinate space $x_1, x_2, \cdots, x_n$; dS denotes a surface element; ν denotes the direction of the outer normal to the surface S. The integration is denoted by a single integral sign regardless of the dimensionality of the integral. These conventions will be used throughout.

2. Classification of Second-order Partial Differential Equations and Their Reduction to Canonical Form

Let us perform a transformation of Equation (1.3) by introducing the new independent variables

$$\xi = \varphi_1(x, y) \qquad \eta = \varphi_2(x, y) \tag{1.14}$$