ORDINARY DIFFERENTIAL EQUATIONS

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PREFACE

The major aims of this book are to provide a text in the spirit of the current emphasis on quality mathematics for all students of the subject and to make available to the student those ideas which are fundamental to the understanding of ordinary differential equations.

The book is planned so that it may be used in different ways, depending on the preparation of the class. Experience in teaching the material indicates that students who have completed a sound course in elementary calculus will be able to understand the statements of all theorems and that they will find the solutions of all but a few exercises attainable. Numerous worked examples are provided. Teachers of this subject are aware that the proofs of some of the central theorems necessarily require a working knowledge of advanced calculus. Accordingly, if it is desired to include in the course all proofs provided by the book, the student should have had such a course, or at least be taking one concurrently.

A word or two about the arrangement of the book may be helpful. The first two chapters deal with the usual elementary methods of solving first-order differential equations and linear differential equations with constant coefficients. The methods are largely formal. Simple applications appear in Chapter 3. The study of the nature of solutions starts in Chapter 4, which begins, as it must, with a careful definition of a solution—a definition that simply formalizes what the student has already learned. The existence theorem is stated, and simple applications are made in this chapter and again in the study of properties of solutions of linear differential equations in Chapters 5 and 7. Its proof appears in Chapter 9. (Here, and at many points, the book follows the time-tested "first how, then why" approach to learning mathematics.) The book contains material on the Laplace transform (Chapter 2, Sections 6, 7, and 8), an additional chapter on applications (Chapter 6), and the ideas involved in Liapunov's "direct method"—ideas that have become of such great importance in recent times in studying the stability of differential systems (Chapter 7, Sections 4, 5, and 6). A special effort has been made to present these results in such a way that they will be usable by the student even though the proofs of the relevant theorems are rather sophisticated. A reasonably adequate account of power-series solutions appears in Chapter 8. The methods of this chapter are, of course, necessarily rather formal. The book concludes with Chapters 10 and 11 on the fundamentals of oscillation theory and the related theory of characteristic functions.

The text is accordingly so arranged that classes unfamiliar with differential equations may begin with Chapter 1. Other classes already acquainted with elementary methods of solving differential equations may review the first three chapters quickly before beginning their study of Chapter 4. And some classes, those for whom the ideas of Chapter 1–3 are quite familiar, may well choose to begin their course of study with Chapter 4. Chapters 3 and 6, the chapters on physical applications, may be omitted by classes interested only in the mathematical theory without interrupting the continuity of the text.

My thanks are owed to many. Helpful ideas have come from many teachers and students who used my earlier book on the same subject. Special thanks are due Professors R. H. Bruck and R. C. MacCamy for their valued comments. I wish to acknowledge helpful and stimulating conversations I have had with Professors S. Lefschetz, Jack Hale, Joseph LaSalle, and their colleagues at RIAS. Finally, I wish to express my gratitude to The McGraw-Hill Book Company for permitting use of material from my earlier book and to the present publishers for their unfailing courtesy and helpfulness to me in the publication of the present book.

WALTER LEIGHTON

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1

Elementary Methods

1 Introduction

Differential equations are equations that involve derivatives. For example, the equations

$$y' = f(x),$$

$$y'' + y = 0,$$

$$(1.1) \qquad y'' = (1 + y'^2)^{1/2},$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

are differential equations. The first three of these equations are called ordinary differential equations because they involve the ordinary derivatives of the unknown y. The last equation is an example of a partial differential equation. We shall be concerned with ordinary differential equations and their solutions.

To solve an algebraic equation, such as

$$(1.2) \qquad x^2 - 3x + 2 = 0,$$

we seek a number with the property that when the unknown x is replaced by this number the left-hand member of the equation reduces to zero. In equation (1.2) either the number 1 or the number 2 has this property. We say that this equation has the two solutions 1 and 2. To solve a differential equation we seek to determine not an unknown number but an unknown function. For example, in the equation

$$y'' + y = 0, \tag{1.3}$$

y is regarded as the unknown. To find a solution we attempt to determine *a function defined on an interval* with the property that when y is replaced by this function, the equation reduces to an identity on this interval. It is clear that $\sin x$ is a solution of (1.3) for all values of x, for,

$$(\sin x)'' + \sin x \equiv 0 \qquad (-\infty < x < \infty).$$

Similarly, it is easy to verify that $\cos x$ is also a solution of the differential equation (1.3).

Differential equations play a fundamental role in almost every branch of science and of engineering. They are of central importance in mathematical analysis. A differential equation describes the flow of current in a conductor; another describes the flow of heat in a slab. Other differential equations describe the motion of an intercontinental missile; still another describes the behavior of a chemical mixture. Sometimes it is important to find a particular solution of a given differential equation. Often we are more interested in the existence and behavior of solutions of a given differential equation than we are in finding its solutions.

In this chapter we shall begin our study by solving certain simple and important types of differential equations.

The order of a differential equation is the order of the highest derivative that appears in the equation. Accordingly, the first equation in (1.1) is of first order, and the next two equations are of second order. Similarly, the differential equation

$$y''' + y^4 = e^x$$

is of the third order, and the equation

$$(y'''')^2 + yy' = 3$$

is of fourth order. The differential equation

$$M(x,y) + N(x,y)y' = 0 \tag{1.4}$$

is of first order. It is frequently useful to rewrite this equation in the form

$$M(x,y)\,dx + N(x,y)\,dy = 0. \tag{1.4$'$}$$

Thus,

$$(x^2 + y^2)\,dx + 2x\,dy = 0,$$
$$xe^y\,dx + (1 + y)\,dy = 0$$

are differential equations of the first order written in the form (1.4)′.

EXERCISES

1. Verify that if c_1 and c_2 are constants, $c_1 \sin x + c_2 \cos x$ is a solution of the differential equation $y'' + y = 0$.

2. Find by inspection a solution of each of the following differential equations:

(a) $y' - y = 0$;
(b) $y' + 2y = 0$;
(c) $y' = \sin x$.

3. Verify that the function $c_1e^x + c_2e^{2x}$ (c_1, c_2 constants) is a solution of the differential equation $y'' - 3y' + 2y = 0$.

4. Determine $r(x)$ such that the function $\sin \log x$ $(x > 0)$ is a solution of the differential equation $[r(x)y']' + \dfrac{y}{x} = 0$.

5. Verify that $\sin x$ is a solution of the differential equation $y'^2 + y^2 = 1$.

6. Verify that if c_1 and c_2 are constants and $x > 0$, the function $c_1 \sin \dfrac{1}{x} + c_2 \cos \dfrac{1}{x}$ is a solution of the differential equation $(x^2y')' + x^{-2}y = 0$.

ANSWERS

2. (a) e^x.

4. x.

2 Exact Differential Equations of First Order

A particularly important class of differential equations are the so-called *exact* differential equations. A differential equation

$$M(x,y)\,dx + N(x,y)\,dy = 0 \tag{2.1}$$

is said to be exact if there exists a function $g(x,y)$ such that

$$d[g(x,y)] = M(x,y)\,dx + N(x,y)\,dy;$$

that is to say, if there exists a function $g(x,y)$ such that

$$g_x(x,y) = M(x,y) \quad \text{and} \quad g_y(x,y) = N(x,y).$$

Thus, the equation

$$(4x - y)\,dx + (2y - x)\,dy = 0 \tag{2.2}$$

is exact, since its left-hand member is the differential of the function,

$$g(x,y) = 2x^2 - xy + y^2,$$

for

$$d(2x^2 - xy + y^2) = (4x - y)\,dx + (2y - x)\,dy.$$

Clearly, we might equally well have chosen $g(x,y) = 2x^2 - xy + y^2 + 3$, or $g(x,y) = 2x^2 - xy + y^2 + c$, where c is any constant.

When $g(x,y)$ is a differentiable function such that

$$d[g(x,y)] = M(x,y)\,dx + N(x,y)\,dy,$$

any function $g(x,y) - c$, where c is a constant, is called an *integral* of the corresponding differential equation (2.1). Curves defined by the equations

$$g(x,y) = c \qquad (c \text{ constant})$$

are called *integral curves* of the differential equation. Accordingly, the function $2x^2 - xy + y^2$ is an integral of equation (2.2). Integral curves of equation (2.2) are given by the equation

$$2x^2 - xy + y^2 = c. \tag{2.3}$$

When $c > 0$ the curves given by (2.3) are readily seen to be ellipses.

It is natural to inquire how we may identify those differential equations (2.1) that are exact, and how, when they are exact, corresponding integrals $g(x,y)$ may be determined. The following theorem is fundamental.

Theorem 2.1. *If the functions $M(x,y)$, $N(x,y)$ and the partial derivatives $M_y(x,y)$, $N_x(x,y)$ are continuous in a square region R, a necessary and sufficient condition that the differential equation*

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

be exact is that

$$\frac{\partial M}{\partial y} \equiv \frac{\partial N}{\partial x}. \tag{2.4}$$

The proof of the necessity of the condition is immediate. We suppose the differential equation is exact; that is, there exists a function $g(x,y)$ with the property that

$$g_x(x,y) = M(x,y), \qquad g_y(x,y) = N(x,y).$$

Since $g_{xy} = g_{yx}$ it follows at once that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

The proof of the necessity is complete.

In proving the sufficiency we exhibit a function $g(x,y)$ whose partial derivatives satisfy the condition

$$g_x(x,y) = M(x,y) \qquad g_y(x,y) = N(x,y).$$

Such a function is*

$$g(x,y) = \int_{x_0}^{x} M(x,y_0)\,dx + \int_{y_0}^{y} N(x,y)\,dy, \tag{2.5}$$

* The student who is familiar with line integrals will recognize the integral in (2.5) as the line integral $\int M\,dx + N\,dy$ taken over an "elbow path" from the point (x_0, y_0) to (x,y). Condition (2.4) will be seen to be precisely the condition that the line integral be independent of the path in R.

where (x_0, y_0) is a fixed point and (x, y) is an arbitrary point of the region R. For,

$$\begin{aligned} g_x &= M(x,y_0) + \int_{y_0}^{y} N_x(x,y)\,dy \\ &= M(x,y_0) + \int_{y_0}^{y} M_y(x,y)\,dy \\ &= M(x,y_0) + [M(x,y) - M(x,y_0)] \\ &= M(x,y); \end{aligned}$$

while

$$g_y = N(x,y).$$

The proof of the theorem is complete.

Example. In the differential equation

$$(3x^2 + y^2)\,dx + 2xy\,dy = 0 \tag{2.6}$$

we see that $M(x,y) = 3x^2 + y^2$, $N(x,y) = 2xy$, and $M_y = 2y$, $N_x = 2y$. Thus, the differential equation is exact. To find an integral $g(x,y)$ we choose the point (x_0, y_0) to be the origin, and we have

$$g(x,y) = \int_0^x (3x^2 + 0^2)\,dx + \int_0^y 2xy\,dy$$

$$= x^3 + xy^2. \tag{2.7}$$

It is easily seen that the differential of (2.7) is given by the left-hand member of (2.6). Integral curves are given by the equation

$$x^3 + xy^2 = c, \tag{2.8}$$

where c is a constant.

By finding (2.8) we have solved equation (2.6) in the sense that if we solve equation (2.8) for y and obtain a differentiable function of x, then that function is a solution of the differential equation. Specifically, we find that

$$y = \pm\sqrt{\frac{1}{x}(c - x^3)}.$$

It is easy to verify that both $\sqrt{\frac{1}{x}(c - x^3)}$ and $-\sqrt{\frac{1}{x}(c - x^3)}$ are indeed solutions of (2.6) over a suitable interval of the x-axis.

This is the general situation, as will be seen from the following theorem.

Theorem 2.2. *If* $g(x,y)$ *is an integral of the exact differential equation* $M(x,y)\,dx + N(x,y)\,dy = 0$, *any differentiable solution* $y(x)$ *of the equation* $g(x,y) = c$ *is a solution of the differential equation.*

To prove the theorem note that because $g(x,y)$ is an integral of the differential equation it follows that

$$g_x(x,y) \equiv M(x,y), \qquad g_y(x,y) \equiv N(x,y).$$

Thus,

$$g_x[x,y(x)] \equiv M[x,y(x)], \qquad g_y[x,y(x)] \equiv N[x,y(x)].$$

Further, since $y(x)$ is a solution of the equation $g(x,y) = c$, we have

$$g[x,y(x)] \equiv c \qquad (c \text{ constant}).$$

It follows that

$$g_x[x,y(x)] + g_y[x,y(x)]y'(x) \equiv 0$$

and, hence, that

$$M[x,y(x)] + N[x,y(x)]y'(x) \equiv 0;$$

that is to say, $y(x)$ is a solution of the given differential equation.

Alternate method. The line integral (2.5) provides a simple and direct method of solving an exact differential equation. An alternate method, the validity of which may be established by the preceding analysis, will be illustrated by an example.

We have observed that the differential equation

$$(3x^2 + y^2)\,dx + 2xy\,dy = 0 \qquad (2.9)$$

is exact. To find an integral we first integrate the term $2xy\,dy$ formally with respect to y, obtaining

$$xy^2.$$

Next, we determine a function $f(x)$, of x alone, such that

$$d[xy^2 + f(x)]$$

is given by the left-hand member of (2.9). That is, we wish to find a function $f(x)$ such that

$$2xy\,dy + y^2\,dx + f'(x)\,dx = (3x^2 + y^2)\,dx + 2xy\,dy.$$

This is equivalent to the equation

$$f'(x) = 3x^2.$$

It follows that

$$f(x) = x^3 + c,$$

and that integrals of (2.9) are given by

$$xy^2 + x^3 + c.$$

We might equally well have commenced by integrating formally the term $(3x^2 + y^2)\,dx$, obtaining $x^3 + xy^2$. Then we seek to determine a function $g(y)$, of y alone, such that $d[x^3 + xy^2 + g(y)]$ is given by the left-hand member of (2.9).

An advantage of the alternate method may be observed in the first treatment of equation (2.9). Clearly, if we can determine the function $f(x)$, the equation is necessarily solved. It is desirable, however, to demonstrate that under the conditions of Theorem 2.1, such a function $f(x)$ can always be determined. This can be seen as follows. Consider the differential equation

$$M(x,y)\,dx + N(x,y)\,dy = 0,$$

and suppose that the conditions of Theorem 2.1 are satisfied. By "formal integration" of the term $N(x,y)\,dy$ is meant determining a function

$$H(x,y) = k + \int_{y_0}^{y} N(x,y)\,dy \qquad (k \text{ constant}),$$

where the points (x, y_0) and (x, y) lie in R. We note that

$$H_y(x,y) = N(x,y),$$

$$(2.10) \qquad H_x(x,y) = \int_{y_0}^{y} N_x(x,y)\,dy = \int_{y_0}^{y} M_y(x,y)\,dy$$

$$= M(x,y)\Big|_{y=y_0}^{y=y} = M(x,y) - M(x,y_0).$$

To complete the demonstration we show that there exists a function $f(x)$, of x alone, such that

$$(2.11) \qquad d[H(x,y) + f(x)] = M(x,y)\,dx + N(x,y)\,dy.$$

This is equivalent to demonstrating that there exists a function $f(x)$ such that

$$H_x(x,y)\,dx + H_y(x,y)\,dy + f'(x)\,dx = M(x,y)\,dx + N(x,y)\,dy,$$

or, by (2.10), such that

$$f'(x) = M(x,y_0).$$

It is clear that $f(x)$ may be taken as

$$f(x) = \int_{x_0}^{x} M(x,y_0)\,dx,$$

if (x_0, y_0) lies in R.

Remark. It is frequently desirable in differential equation theory to note a distinction between solving a differential equation and finding a solution of a differential equation. Recall that a *solution* is always a function of x defined on an interval which satisfies the differential equation. On the other hand, it is customary to regard a first-order differential equation as *solved* when we can write equations of its integral curves. Theorem 2.2, of course, justifies this seeming ambiguity.