

David Gilbarg Neil S. Trudinger

Elliptic Partial
Differential Equations
of Second Order

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Preface

This volume is intended as an essentially self contained exposition of portions of the theory of second order quasilinear elliptic partial differential equations, with emphasis on the Dirichlet problem in bounded domains. It grew out of lecture notes for graduate courses by the authors at Stanford University, the final material extending well beyond the scope of these courses. By including preparatory chapters on topics such as potential theory and functional analysis, we have attempted to make the work accessible to a broad spectrum of readers. Above all, we hope the readers of this book will gain an appreciation of the multitude of ingenious barehanded techniques that have been developed in the study of elliptic equations and have become part of the repertoire of analysis.

Many individuals have assisted us during the evolution of this work over the past several years. In particular, we are grateful for the valuable discussions with L. M. Simon and his contributions in Sections 15.4 to 15.8; for the helpful comments and corrections of J. M. Cross, A. S. Geue, J. Nash, P. Trudinger and B. Turkington; for the contributions of G. Williams in Section 10.5 and of A. S. Geue in Section 10.6; and for the impeccably typed manuscript which resulted from the dedicated efforts of Isolde Field at Stanford and Anna Zalucki at Canberra. The research of the authors connected with this volume was supported in part by the National Science Foundation.

August 1977

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Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen mit besonderer Berücksichtigung der Anwendungsgebiete

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Chapter 1

Introduction

Summary

The principal objective of this work is the systematic development of the general theory of second order quasilinear elliptic equations and of the linear theory required in the process. This means we shall be concerned with the solvability of boundary value problems (primarily the Dirichlet problem) and related general properties of solutions of linear equations

$$(1.1) \quad Lu \equiv a^{ij}(x)D_{ij}u + b^i(x)D_i u + c(x)u = f(x), \quad i, j = 1, 2, \dots, n.$$

and of quasilinear equations

$$(1.2) \quad Qu \equiv a^{ij}(x, u, Du)D_{ij}u + b(x, u, Du) = 0.$$

Here $Du = (D_1 u, \dots, D_n u)$, where $D_i u = \partial u / \partial x_i$, $D_{ij} u = \partial^2 u / \partial x_i \partial x_j$, etc., and the summation convention is understood. The ellipticity of these equations is expressed by the fact that the coefficient matrix $[a^{ij}]$ is (in each case) positive definite in the domain of the respective arguments. We refer to an equation as *uniformly elliptic* if the ratio γ of maximum to minimum eigenvalue of the matrix $[a^{ij}]$ is bounded. We shall be concerned with both non-uniformly and uniformly elliptic equations.

The classical prototypes of linear elliptic equations are of course Laplace's equation

$$\Delta u = \sum D_{ii} u = 0$$

and its inhomogeneous counterpart, Poisson's equation $\Delta u = f$. Probably the best known example of a quasilinear elliptic equation is the minimal surface equation

$$\sum D_i (D_i u / (1 + |Du|^2)^{1/2}) = 0,$$

which arises in the problem of least area. This equation is non-uniformly elliptic, with $\gamma = 1 + |Du|^2$. The properties of the differential operators in these examples motivate much of the theory of the general classes of equations discussed in this book.

The relevant linear theory is developed in Chapters 2–8 (and in part of Chapter 11). Although this material has independent interest, the emphasis here is on aspects needed for application to nonlinear problems. Thus the theory stresses weak hypotheses on the coefficients and passes over many of the important classical and modern results on linear elliptic equations.

Since we are ultimately interested in classical solutions of equation (1.2), what is required at some point is an underlying theory of classical solutions for a sufficiently large class of linear equations. This is provided by the Schauder theory in Chapter 6, which is an essentially complete theory for the class of equations (1.1) with Hölder continuous coefficients. Whereas such equations enjoy a definitive existence and regularity theory for classical solutions, corresponding results cease to be valid for equations in which the coefficients are assumed only continuous.

A natural starting point for the study of classical solutions is the theory of Laplace's and Poisson's equations. This is the content of Chapters 2 and 4. In anticipation of later developments the Dirichlet problem for harmonic functions with continuous boundary values is approached through the Perron method of subharmonic functions. This emphasizes the maximum principle, and with it the barrier concept for studying boundary behavior, in arguments that are readily extended to more general situations in later chapters. In Chapter 4 we derive the basic Hölder estimates for Poisson's equation from an analysis of the Newtonian potential. The principal result here (see Theorems 4.6, 4.8) states that all $C^2(\Omega)$ solutions of Poisson's equation, $\Delta u = f$, in a domain Ω of \mathbb{R}^n satisfy a uniform estimate in any subset $\Omega' \subset \subset \Omega$

$$(1.3) \quad \|u\|_{C^{2,\alpha}(\Omega')} \leq C(\sup_{\Omega} |u| + \|f\|_{C^{\alpha}(\bar{\Omega})}),$$

where C is a constant depending only on α ($0 < \alpha < 1$), the dimension n and $\text{dist}(\Omega', \partial\Omega)$; (for notation see Section 4.1). This *interior estimate* (interior since $\Omega' \subset \subset \Omega$) can be extended to a *global estimate* for solutions with sufficiently smooth boundary values provided the boundary $\partial\Omega$ is also sufficiently smooth. In Chapter 4 estimates up to the boundary are established only for hyperplane and spherical boundaries, but these suffice for the later applications.

The climax of the theory of classical solutions of linear second order elliptic equations is achieved in the Schauder theory, which is developed in modified and expanded form in Chapter 6. Essentially, this theory extends the results of potential theory to the class of equations (1.1) having Hölder continuous coefficients. This is accomplished by the simple but fundamental device of regarding the equation locally as a perturbation of the constant coefficient equation obtained by fixing the leading coefficients at their values at a single point. A careful calculation based on the above mentioned estimates for Poisson's equation yields the same inequality (1.3) for any $C^{2,\alpha}$ solution of (1.1), where the constant C now depends also on the bounds and Hölder constants of the coefficients and in addition on the minimum and maximum eigenvalues of the coefficient matrix $[a^{ij}]$ in Ω . These results are stated as interior estimates in terms of weighted interior norms (Theorem 6.2) and, in the case of sufficiently smooth boundary data, as global estimates in terms of global norms (Theorem 6.6). Here we meet the important and recurring concept of

an *a priori* estimate; namely, an estimate (in terms of given data) valid for all *possible* solutions of a class of problems even if the hypotheses do not guarantee the existence of such solutions. A major part of this book is devoted to the establishment of *a priori* bounds for various problems. (We have taken the liberty of replacing the latin *a priori* with the single word *a priori*, which will be used throughout.)

The importance of such *a priori* estimates is visible in several applications in Chapter 6, among them in establishing the solvability of the Dirichlet problem by the method of continuity (Theorem 6.8) and in proving the higher order regularity of C^2 solutions under appropriate smoothness hypotheses (Theorems 6.17, 6.19). In both cases the estimates provide the necessary compactness properties for certain classes of solutions, from which the desired results are easily inferred.

We remark on several additional features of Chapter 6, which are not needed for the later developments but which broaden the scope of the basic Schauder theory. In Section 6.5 it is seen that for continuous boundary values and a suitably wide class of domains the proof of solvability of the Dirichlet problem for (1.1) can be achieved entirely with interior estimates, thereby simplifying the structure of the theory. The results of Section 6.6 extend the existence theory for the Dirichlet problem to certain classes of non-uniformly elliptic equations. Here we see how relations between geometric properties of the boundary and the degenerate ellipticity at the boundary determine the continuous assumption of boundary values. The methods are based on barrier arguments that foreshadow analogous (but deeper) results for nonlinear equations in Part II. In Section 6.7 we extend the theory of (1.1) to the regular oblique derivative problem. The method is basically an extrapolation to these boundary conditions of the earlier treatment of Poisson's equation and the Schauder theory (without barrier arguments, however).

In the preceding considerations, especially in the existence theory and barrier arguments, the maximum principle for the operator L (when $c \leq 0$) plays an essential part. This is a special feature of second order elliptic equations that simplifies and strengthens the theory. The basic facts concerning the maximum principle, as well as illustrative applications of comparison methods, are contained in Chapter 3. The maximum principle provides the earliest and simplest *a priori* estimates of the general theory. It is of considerable interest that all the estimates of Chapters 4 and 6 can be derived entirely from comparison arguments based on the maximum principle, without any mention of the Newtonian potential or integrals.

An alternative and more general approach to linear problems, without potential theory, can be achieved by Hilbert space methods based on *generalized* or *weak* solutions, as in Chapter 8. To be more specific, let L' be a second order differential operator, with principal part of *divergence form*, defined by

$$L'u \equiv D_i(a^{ij}(x)D_j u + b^i(x)u) + c^i(x)D_i u + d(x)u.$$

If the coefficients are sufficiently smooth, then clearly this operator falls within the class discussed in Chapter 6. However, even if the coefficients are in a much wider class and u is only weakly differentiable (in the sense of Chapter 7), one can still define weak or generalized solutions of $L'u = g$ in appropriate function classes.

In particular, if the coefficients a^{ij} , b^i , c^i are bounded and measurable in Ω and g is an integrable function in Ω , let us call u a weak or generalized solution of $L'u = g$ in Ω if $u \in W^{1,2}(\Omega)$ (as defined in Chapter 7) and

$$(1.4) \quad \int_{\Omega} [(a^{ij} D_j u + b^i u) D_i v - (c^i D_i u + du) v] dx = - \int_{\Omega} g v dx$$

for all test functions $v \in C_0^1(\Omega)$. It is clear that if the coefficients and g are sufficiently smooth and $u \in C^2(\Omega)$, then u is also a classical solution.

We can now speak also of weak solutions u of the *generalized Dirichlet problem*,

$$L'u = g \text{ in } \Omega, \quad u = \varphi \text{ on } \partial\Omega,$$

if u is a weak solution satisfying $u - \varphi \in W_0^{1,2}(\Omega)$, where $\varphi \in W^{1,2}(\Omega)$. Assuming that the minimum eigenvalue of $[a^{ij}]$ is bounded away from zero in Ω , that

$$(1.5) \quad D_i b^i + d \leq 0$$

in the weak sense, and that also $g \in L^2(\Omega)$, we find in Theorem 8.3 that the generalized Dirichlet problem has a unique solution $u \in W^{1,2}(\Omega)$. Condition (1.5), which is the analogue of $c \leq 0$ in (1.1), assures a maximum principle for weak solutions of $L'u \geq 0$ (≤ 0) (Theorem 8.1) and hence uniqueness for the generalized Dirichlet problem. Existence of a solution then follows from the Fredholm alternative for the operator L' (Theorem 8.6), which is proved by an application of the Riesz representation theorem in the Hilbert space $W_0^{1,2}(\Omega)$.

The major part of Chapter 8 is taken up with the regularity theory for weak solutions. Additional regularity of the coefficients in (1.4) implies that the solutions belong to higher $W^{k,2}$ spaces (Theorems 8.8, 8.10). It follows from the Sobolev imbedding theorems in Chapter 7 that weak solutions are in fact classical solutions provided the coefficients are sufficiently regular. Global regularity of these solutions are inferred by extending interior regularity to the boundary when the boundary data are sufficiently smooth (Theorems 8.13, 8.14).

The regularity theory of weak solutions and the associated pointwise estimates are fundamental to the nonlinear theory. These results provide the starting point for the "bootstrap" arguments that are typical of nonlinear problems. Briefly, the idea here is to start with weak solutions of a quasilinear equation, regarding them as weak solutions of related *linear* equations obtained by inserting them into the coefficients, and then to proceed by establishing improved regularity of these solutions. Starting anew with the latter solutions and repeating the process, still further regularity is assured, and so on, until the original weak solutions are finally proved to be suitably smooth. This is the essence of the regularity proofs for the older variational problems and is implicit in the nonlinear theory presented here.

The Hölder estimates for weak solutions that are so vital for the nonlinear theory are derived in Chapter 8 from Harnack inequalities based on the Moser iteration technique (Theorems 8.17, 8.18, 8.20, 8.24). These results generalize the basic a priori Hölder estimate of De Giorgi, which provided the initial breakthrough in the theory of quasilinear equations in more than two independent variables.

The arguments rest on integral estimates for weak solutions u derived from judicious choice of test functions v in (1.4). The test function technique is the dominant theme in the derivation of estimates throughout most of this work.

Part II of this book is devoted largely to the Dirichlet problem and related estimates for quasilinear equations. The results concern in part the general operator (1.2) while others apply especially to operators of divergence form

$$(1.6) \quad Qu \equiv \operatorname{div} A(x, u, Du) + B(x, u, Du)$$

where $A(x, z, p)$ and $B(x, z, p)$ are respectively vector and scalar functions defined on $\Omega \times \mathbb{R} \times \mathbb{R}^n$.

Chapter 9 extends maximum and comparison principles (analogous to results in Chapter 3) to solutions and subsolutions of quasilinear equations. We mention in particular apriori bounds for solutions of $Qu \geq 0$ ($=0$), where Q is a divergence form operator satisfying certain structure conditions more general than ellipticity (Theorem 9.7).

Chapter 10 provides the basic framework for the solution of the Dirichlet problem in the following chapters. We are concerned principally with classical solutions, and the equations may be uniformly or non-uniformly elliptic. Under suitable general hypotheses any globally smooth solution u of the boundary value problem for $Qu=0$ in a domain Ω with smooth boundary can be viewed as a fixed point, $u=Tu$, of a compact operator T from $C^{1,\alpha}(\bar{\Omega})$ to $C^{1,\alpha}(\bar{\Omega})$ for any $\alpha \in (0,1)$. In the applications the function defined by Tu , for any $u \in C^{1,\alpha}(\bar{\Omega})$, is the unique solution of the *linear* problem obtained by inserting u into the coefficients of Q . The Leray-Schauder fixed point theorem (proved in Chapter 10) then implies the existence of a solution of the boundary value problem provided an apriori bound, in $C^{1,\alpha}(\bar{\Omega})$, can be established for the solutions of a related continuous family of equations $u=T(u; \sigma)$, $0 \leq \sigma \leq 1$, where $T(u; 1)=Tu$ (Theorems 10.4, 10.7). The establishment of such bounds for certain broad classes of Dirichlet problems is the object of Chapter 12–14.

The general procedure for obtaining the required apriori bound for possible solutions u is a four-step process involving successive estimation of $\sup_{\Omega} |u|$, $\sup_{\partial\Omega} |Du|$, $\sup_{\Omega} |Du|$, and $\|u\|_{C^{1,\alpha}(\bar{\Omega})}$ for some $\alpha > 0$. Each of these estimates presupposes the preceding ones and the final bound on $\|u\|_{C^{1,\alpha}(\bar{\Omega})}$ completes the existence proof based on the Leray-Schauder theorem.

As already observed, bounds on $\sup_{\Omega} |u|$ are discussed in Chapter 9. In the later chapters this bound is either assumed in the hypotheses or is implied by properties of the equation.

Equations in two variables (Chapter 11) occupy a special place in the theory. This is due in part to the distinctive methods that have been developed for them and also to the results, some of which have no counterpart for equations in more than two variables. The method of quasiconformal mappings and arguments based on divergence structure equations (cf. Chapter 10) are both applicable to equations in two variables and yield relatively easily the desired $C^{1,\alpha}$ apriori estimates, from which a solution of the Dirichlet problem follows readily.

Of particular interest is the fact that solutions of uniformly elliptic linear equations in two variables satisfy an a priori $C^{1,\alpha}$ estimate depending only on the ellipticity constants and bounds on the coefficients, without any regularity assumptions (Theorem 11.4). Such a $C^{1,\alpha}$ estimate, or even the existence of a gradient bound under the same general conditions is unknown for equations in more than two variables. Another special feature of the two-dimensional theory is the existence of an a priori C^1 bound $|Du| \leq K$ for solutions of arbitrary elliptic equations

$$(1.7) \quad au_{xx} + 2bu_{xy} + cu_{yy} = 0,$$

where u is continuous on the closure of a bounded convex domain Ω and has boundary value φ on $\partial\Omega$ satisfying a bounded slope (or three-point) condition with constant K . This classical result, usually based on a theorem of Radó on saddle surfaces, is given an elementary proof in Lemma 11.6. The stated gradient bound, which is valid for all solutions u of the general quasilinear equation (1.7) in which $a = a(x, y, u, u_x, u_y)$, etc., and such that $u = \varphi$ on $\partial\Omega$, reduces this Dirichlet problem to the case of uniformly elliptic equations treated in Theorem 11.5. In Theorem 11.7 we obtain a solution of the general Dirichlet problem for (1.7), assuming local Hölder continuity of the coefficients and a bounded slope condition for the boundary data (without further smoothness restrictions on the data).

Chapters 12, 13 and 14 are devoted to the derivation of the gradient estimates involved in the existence procedure described above. In Chapter 12, we prove the fundamental results of Ladyzhenskaya and Ural'tseva on Hölder estimates of derivatives of elliptic quasilinear equations. In Chapter 13 we study the estimation of the gradient of solutions of elliptic quasilinear equations on the boundary. After considering general and convex domains, we give an account of the theory of Serrin which associates generalized boundary curvature conditions with the solvability of the Dirichlet problem. In particular, we are able to conclude from the results of Chapters 10, 12 and 13 the Jenkins and Serrin criterion for solvability of the Dirichlet problem for the minimal surface equation, namely, that this problem is solvable for smooth domains and arbitrary smooth boundary values if and only if the mean curvature of the boundary (with respect to the inner normal) is non-negative at every point (Theorem 13.14).

Global and interior gradient bounds for solutions u of quasilinear equations are established in Chapter 14. Following a refinement of an old procedure of Bernstein we derive estimates for $\sup_{\Omega} |Du|$ in terms of $\sup_{\partial\Omega} |Du|$ for classes of equations that include both uniformly elliptic equations satisfying natural growth conditions and equations sharing common structural properties with the prescribed mean curvature equation (Theorem 14.2). A variant of our approach yields interior gradient estimates for a more restricted class of equations (Theorem 14.3). We also consider uniformly and non-uniformly elliptic equations in divergence form (Theorems 14.4, 14.5 and 14.6), in which cases, by appropriate test function arguments, we deduce gradient estimates under different types of coefficient conditions than in the general case. We conclude Chapter 14 with a selection of existence theorems, chosen to illustrate the scope of the theory. These theorems are all obtained by various combinations of the a priori estimates in Chapters 9, 13 and 14

and a judicious choice of a related family of problems to which Theorem 10.8 can be applied.

In Chapter 15, we concentrate on the prescribed mean curvature equation and derive an interior gradient bound (Theorem 15.5) thereby enabling us to deduce existence theorems for the Dirichlet problem when only continuous boundary values are assigned (Theorems 15.8, 15.10). We also consider a family of equations in two variables, which in a certain sense bear the same relationship to the prescribed mean curvature equation as the uniformly elliptic equations of Chapter 11 bear to Laplace's equation. Indeed, by means of a generalized notion of quasiconformal mapping, we derive interior estimates for first and second derivatives. The second derivative estimates provide a generalization of a well known curvature estimate of Heinz for solutions of the minimal surface equation (Theorem 15.20) and moreover, imply an extension of the famous result of Bernstein that entire solutions of the minimal surface equation in two variables must be linear (Corollary 15.19). However, perhaps the striking feature of Theorems 15.5 and 15.20 is the approach. Rather than working in the domain Ω , we work on the hypersurface S given by the graph of the solution u and exploit various relations between the tangential gradient and Laplacian operators on S and the mean curvature of S .

We conclude this summary with some guides to the reader. The material is not in strict logical order. Thus the theory of Poisson's equation (Chapter 4) would normally follow Laplace's equation (Chapter 2). However, the elementary character of the results on the maximum principle (Chapter 3) and the opportunity for the reader to meet early some general problems with variable coefficients recommends its insertion after Chapter 2. In fact, the general maximum principle is not used until the existence theory of Chapter 6. The basic material on functional analysis (Chapter 5) is needed in only a minor way for the Schauder theory: the contraction mapping principle and the basic concepts of Banach spaces suffice, except for the proof of the alternative in Theorem 6.15. For applications to nonlinear problems in Part II it is sufficient to know the results of Section 1-3 of Chapter 6. Depending on the reader's interests, it may be preferable to study the linear theory by starting directly with L^2 theory in Chapter 8; this assumes the preliminary material on functional analysis (Chapter 5) and on the calculus of weakly differentiable functions (Chapter 7). The Harnack inequalities and Hölder estimates in the regularity theory of Chapter 8 are not applied until Chapter 12.

The theory of quasilinear equations in two variables (Chapter 11) is essentially independent of Chapters 7-10 and can be read following Chapter 6 provided one assumes the Schauder fixed point theorem (Theorem 10.1). The method of quasiconformal mappings is met again in Chapter 15 but otherwise the remaining chapters are independent of Chapter 11. Accordingly, after the basic outline of the nonlinear theory in Chapter 10 the reader can proceed directly to the n -variable theory in Chapters 12-15. Chapter 15 is largely independent of Chapters 12-14.

Further Remarks

Beyond the assumption of basic real analysis and linear algebra the material in this work is almost entirely self-contained. Thus, much of the preliminary development