INTERACTIVE MATHEMATICS

Topic EII
Essentials of Algebra

Topic 9
Rational Exponents and Radicals

Topic 10
Quadratic Equations

PERSONAL
ACADEMIC
NOTEBOOK

ACADEMIC SYSTEMS

INTERACTIVE MATHEMATICS

Topic EII
Essentials of Algebra

Topic 9
Rational Exponents and Radicals

Topic 10
Quadratic Equations

PERSONAL
ACADEMIC
NOTEBOOK

ACADEMIC
SYSTEMS

Interactive Mathematics

Academic Systems Corporation
444 Castro Street, Suite 1200
Mountain View, CA 94041

800.694.6850

07/98

TABLE OF CONTENTS

LESSON EII.A – REAL NUMBERS AND EXPONENTS

OVERVIEW

Here's what you'll learn in this lesson:

Real Numbers and Notation

a. Number line and notation

b. Operations on signed numbers

c. Properties of real numbers

Integer Exponents

a. Nonnegative exponents

b. Properties of exponents

In algebra you use many numbers, symbols, operations and properties to simplify expressions. Once you are familiar with these basic tools, you can use algebra to solve many different types of problems.

EXPLAIN

REAL NUMBERS AND NOTATION

Summary

An Introduction to Real Numbers and the Number Line

The first numbers were used for counting things, like the number of sheep in a flock or the number of members in a family. This type of number is now called a counting number or natural number. Examples are: 1, 2, 3, 4, 5, ...

As trade and the use of money developed, the idea of zero and negative numbers became necessary to record losses or gains. For this sort of work the integers are the most useful. Examples are: ..., –4, –3, –2, –1, 0, 1, 2, 3, ...

However, there are many everyday quantities which are not measured in whole numbers. You might go to the store and buy a half gallon of juice, or three and a half pounds of cheese, or three quarters of a yard of fabric. This uses yet another type of number called a rational number or fraction. Fractions are written using two whole numbers. For example, $\frac{3}{8}$ and $\frac{11}{29}$ are both fractions.

Here are some examples of sets of numbers:

Counting numbers	1, 2, 3, 4, 5, 6, 7, ...
Whole numbers	0, 1, 2, 3, 4, 5, 6, 7, ...
Integers	..., – 4, – 3, – 2, – 1, 0, 1, 2, 3, 4, 5, ...
Rational numbers	Numbers which can be written as a fraction $\frac{a}{b}$, where a and b are integers and $b \neq 0$. Counting numbers, whole numbers, and integers are all rational numbers. Some examples are: $\frac{7}{15}, -\frac{4}{27}, \frac{13}{1}$
Irrational numbers	Numbers which cannot be written as a fraction $\frac{a}{b}$, where a and b are integers and $b \neq 0$. Some examples are π, $\sqrt{3}$, $\sqrt{11}$

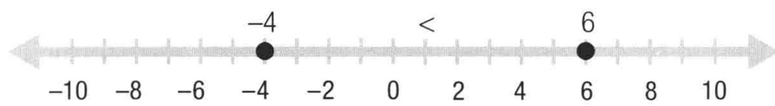
The rational numbers and irrational numbers taken together are the real numbers. Real numbers can be represented as points on a number line.

Comparison Symbols

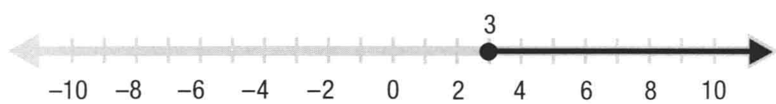
The signs =, ≠, <, >, ≤ or ≥ are used to compare and order numbers.

Symbol	Meaning
=	equal to
≠	not equal to
<	less than
>	greater than
≤	less than or equal to
≥	greater than or equal to

For example, $-4 < 6$ means that the number -4 is less than the number 6. That is, -4 lies to the left of 6 on the number line.

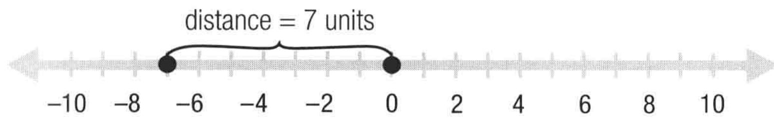


Similarly, $x \geq 3$ means that the variable x can be any real number greater than 3 or equal to 3.



Absolute Value

The absolute value of a number gives the distance of that number from zero on the number line. For example, $|-7|$ denotes that –7 is 7 units from zero on the number line.



Vertical bars enclosing a number are used to denote the absolute value of that number. Absolute value bars leave positive or zero numbers alone but change the sign of negative numbers. So the absolute value of a number is never negative.

For example: $|15| = 15$

$|0| = 0$

$|-7| = 7$

Grouping Symbols

Parentheses or brackets are often used to group symbols together. Grouping symbols determine the order in which operations are performed.

Here's an example: $[7 - 2 \cdot (5 + 13)]$

Operations on Signed Numbers

You can combine positive numbers using the operations of addition, subtraction, multiplication and division. For example:

$2 + 18 = 20$ $\qquad$ $\frac{5}{6} - \frac{1}{4} = \frac{7}{12}$

$13 \cdot 5 = 65$ $\qquad$ $36 \div 3 = 12$

You can also add, subtract, multiply and divide positive **and** negative numbers using operations on signed numbers.

Adding Signed Numbers

To add signed numbers:

If both numbers are positive, their sum is positive.	$2 + 7 = 9$
If both numbers are negative, their sum is negative.	$-3 + (-4) = -7$
If the numbers have different signs, ignore their signs and subtract the smaller number from the larger number. The sum has the same sign as the larger number.	$-3 + 5 = 2$ $3 + (-5) = -2$

Subtracting Signed Numbers

To subtract signed numbers:

1. Change the subtraction sign (–) to an addition sign (+) and change the sign of the number being subtracted.

2. Add according to the rules for addition of signed numbers.

For example, to find $3 - 8$:

1. Change – to + and change the sign of the number being subtracted. $\qquad 3 + (-8)$

2. Add according to the rules for addition of signed numbers. $\qquad = -5$

So, $3 - 8 = -5$.

Multiplying Signed Numbers

To multiply signed numbers:

If the numbers have the same sign, their product is positive.	positive · positive = positive negative · negative = positive	$3 \cdot 5 = 15$ $(-3) \cdot (-5) = 15$
If the numbers have different signs, their product is negative.	positive · negative = negative negative · positive = negative	$3 \cdot (-5) = -15$ $(-3) \cdot 5 = -15$

Dividing Signed Numbers

To divide signed numbers, use the same rules for signs as you used for multiplication:

If the numbers have the same sign, their quotient is positive.	positive ÷ positive = positive negative ÷ negative = positive	$20 \div 4 = 5$ $(-20) \div (-4) = 5$
If the numbers have different signs, their quotient is negative.	positive ÷ negative = negative negative ÷ positive = negative	$20 \div (-4) = -5$ $(-20) \div 4 = -5$

Exponents

Exponents are used to indicate repeated multiplication of the same number.

For example:

exponent

$$2^6 = \underbrace{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2}_{\text{6 factors}} = 64$$

base

In this case, the number 2 is called the base and the number 6 is called the exponent. The exponent 6 indicates that there are 6 factors of 2.

Properties of Real Numbers

On the following page are some properties that will help you simplify calculations with real numbers.

Property	Examples	General Rule (a, b, and c are real numbers)
Commutative Property of Addition	$2 + 8 = 8 + 2$	$a + b = b + a$
Commutative Property of Multiplication	$4 \cdot (-8) = (-8) \cdot 4$	$a \cdot b = b \cdot a$
Associative Property of Addition	$(2 + 8) + 3 = 2 + (8 + 3)$	$(a + b) + c = a + (b + c)$
Associative Property of Multiplication	$(2 \cdot 7) \cdot 3 = 2 \cdot (7 \cdot 3)$	$(a \cdot b) \cdot c = a \cdot (b \cdot c)$
Distributive Property	$6 \cdot (3 + 9) = 6 \cdot 3 + 6 \cdot 9$	$a \cdot (b + c) = a \cdot b + a \cdot c$
Additive Identity Multiplicative Identity	$33 + 0 = 33$ $39 \cdot 1 = 39$	$a + 0 = a$ $a \cdot 1 = a$
Additive Inverse Multiplicative Inverse	$27 + (-27) = 0$ $15 \cdot \frac{1}{15} = 1$	$a + (-a) = 0$ $a \cdot \frac{1}{a} = 1$

Order of Operations

When you are simplifying expressions that involve several operations, you must do them in the correct order. Here is the order you should use:

1. Perform all operations inside parentheses or brackets.
2. Simplify terms with exponents.
3. Multiply or divide, working from left to right.
4. Add or subtract, working from left to right.

Here is a simple example. To find $6 + 2 \cdot 4$:

1. Multiply.	$= 6 + \mathbf{8}$
2. Add.	$= \mathbf{14}$

Here's another example. To simplify the expression $2 - 3^2 \div 3 - (8 - 5) \cdot 4$:

1. Work within the parentheses.	$= 2 - 3^2 \div 3 - \mathbf{3} \cdot 4$
2. Simplify the term with the exponent.	$= 2 - \mathbf{9} \div 3 - 3 \cdot 4$
3. Multiply and divide from left to right.	$= 2 - \mathbf{3} - \mathbf{12}$
4. Add and subtract from left to right.	$= \mathbf{-1} - 12$
	$= \mathbf{-13}$

*One way to recall the order of operations is to remember the phrase "**P**lease **E**xcuse **M**y **D**ear **A**unt **S**ally." The first letter of each word corresponds to an operation – **P**arentheses, **E**xponents, **M**ultiply or **D**ivide, **A**dd or **S**ubtract. However, be careful. This does **not** tell you to multiply before you divide or to add before you subtract.*

Answers to Sample Problems

b. $\frac{1}{13}, -\frac{1}{2}$

c. $-5, 0, \frac{1}{13}, 2, \sqrt{9}, -\frac{1}{2}$

a. 24

b. −24

a. 56

b. −56

b. 35 − 20
15

b. 6

c. 10

Sample Problems

1. From the list below, pick the numbers that are rational numbers.

   $-5, 0, \frac{1}{13}, \sqrt{7}, 2, \sqrt{9}, -\frac{1}{2}$

   ☑ a. Integers are rational numbers. List all of the integers. — $-5, 0, 2, \sqrt{9}$

   (Since $\sqrt{9} = 3$, it too is an integer.)

   ☐ b. Fractions where the numerator and denominator are integers are also rational numbers. List all such fractions. — ________

   ☐ c. List all of the rational numbers. — ________

2. Find: −11 + (−13)

   ☐ a. Ignore the signs and add the two numbers. — $11 + 13 =$ ______

   ☐ b. The answer has a negative sign. — $=$ ______

3. Find: (−7) · (8)

   ☐ a. Ignore the signs and multiply the numbers. — $7 \cdot 8 =$ ______

   ☐ b. Write the answer with its correct sign. — $=$ ______

4. Use the Distributive Property to calculate: $5 \cdot (7 - 4)$

   ☑ a. Use the Distributive Property. — $= 5 \cdot 7 - 5 \cdot 4$

   ☐ b. Simplify. — $=$ ________
   $=$ ________

5. Find: $4 - 2 \cdot (6 - 9)$

   ☑ a. Subtract inside the parentheses. — $= 4 - 2 \cdot (-3)$

   ☐ b. Multiply. — $= 4 +$ ______

   ☐ c. Add. — $=$ ______

INTEGER EXPONENTS

Summary

Nonnegative Integer Exponents

Exponential notation is a shorthand way of writing repeated multiplication.

For example: $3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 = 3^7$

Here, 3 is called the base and 7 is called the exponent or power.

Here is a similar example that uses a variable: $x^5 = x \cdot x \cdot x \cdot x \cdot x$

Here, x is the base and 5 is the exponent or power.

More generally: $x^n = \underbrace{x \cdot x \cdot x \cdot \ldots \cdot x \cdot x}_{n \text{ copies of } x}$

Here, n is a positive integer.

Properties of Exponents

There are several properties which make it easier for you to simplify, multiply, and divide expressions that contain exponents. These are listed below.

Property	Example
Multiplication: $x^m \cdot x^n = x^{m+n}$	$4^3 \cdot 4^7$ $= 4^{3+7}$ $= 4^{10}$
Division: $\frac{x^m}{x^n} = x^{m-n}$ (Here, $x \neq 0$)	$\frac{12^8}{12^5}$ $= 12^{8-5}$ $= 12^3$
Power of a Power: $(x^m)^n = x^{m \cdot n}$	$(7^4)^3$ $= 7^{4 \cdot 3}$ $= 7^{12}$
Power of a Product: $(x \cdot y)^n = x^n \cdot y^n$	$(5 \cdot 8)^9$ $= 5^9 \cdot 8^9$
Power of a Quotient: $\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$	$\left(\frac{5}{11}\right)^4$ $= \frac{5^4}{11^4}$
Zero Exponent: $x^0 = 1$	$9^0 = 1$

To multiply expressions with the same base, add the exponents.

To divide expressions with the same base, subtract the bottom exponent from the top exponent.

To raise a power to a power, multiply the exponents.

To raise a product to a power, raise each factor to that power.

To raise a fraction to a power, raise the numerator and denominator to that power.

Remember, when you see a zero exponent (and the base is not zero), the answer is one.

Be careful, though. There are some cases where the properties of exponents do not apply.

For example, to simplify $5^4 \cdot 7^2$ you cannot use the Multiplication Property, because the two expressions have different bases. Instead, you have to multiply everything out:

$$5^4 \cdot 7^2$$
$$= (5 \cdot 5 \cdot 5 \cdot 5) \cdot (7 \cdot 7)$$
$$= 30{,}625$$

Here is another example. To simplify $3^4 - 3^5$ you can multiply everything out and then add. This is because even though the base is the same, there is no exponent property for addition.

$$3^4 - 3^5$$
$$= 81 - 243$$
$$= -162$$

You can also use the Distributive Property.

$$3^4 - 3^5$$
$$= 3^4(1 - 3)$$
$$= 3^4(-2)$$
$$= 81(-2)$$
$$= -162$$

Sample Problems

1. Calculate: 3^4

   ☑ a. Write the base. The base is 3.

   ☐ b. Write the exponent. ________

   ☐ c. Write the answer. ________

2. Simplify: $(2xy)^4 \cdot x^{12} \cdot y^8$

   ☑ a. First use the Power of a Product rule. $= 2^4 \cdot x^4 \cdot y^4 \cdot x^{12} \cdot y^8$

   ☐ b. Use the Multiplication Property for the factors with base *x*. = ______

   ☐ c. Use the Multiplication Property for the factors with base *y*. = ______

3. Simplify: $\left(\frac{3}{5}\right)^7 \cdot 3^4 \cdot 5^2$

   ☑ a. First use the Power of a Quotient property. $= \frac{3^7}{5^7} \cdot 3^4 \cdot 5^2$

   $= \frac{3^7 \cdot 3^4 \cdot 5^2}{5^7}$

   ☐ b. Use the Multiplication Property for the factors with base 3. = ______

   ☐ c. Use the Division Property for the factors with base 5. = ______

Answers to Sample Problems

b. The exponent is 4.

c. $3 \cdot 3 \cdot 3 \cdot 3 = 81$

b. $2^4 \cdot x^{16} \cdot y^4 \cdot y^8$

c. $2^4 \cdot x^{16} \cdot y^{12}$ *or* $16x^{16}y^{12}$

b. $\frac{3^{11} \cdot 5^2}{5^7}$

c. $\frac{3^{11}}{5^5}$ *or* $3^{11}5^{-5}$

HOMEWORK

Homework Problems

Circle the homework problems assigned to you by the computer, then complete them below.

Explain

Real Numbers and Notation

1. Circle all the inequality symbols that could replace the question mark to make this statement true: $-4 \ ? \ 3$

   $<$ $\leq$ $=$

   $>$ $\geq$ $\neq$

2. Find: $\frac{-45}{9}$

3. Rewrite using the Commutative Property: $17 + 8$

4. Circle the number that is **not** an integer:
   $0, -\frac{1}{2}, \sqrt{16}, \frac{4}{2}, -77$

5. Find: $-7 - (-3)$

6. Rewrite using the Associative Property: $5 \cdot (11 \cdot 23)$

7. Find: $|-9|$

8. Find: $2 - 5 - (-8)$

9. Find the additive inverse of 7 and the multiplicative inverse of 7.

10. Circle all the inequality symbols that could replace the question mark to make this statement true: $|23| \ ? \ -|-23|$

    $<$ $\leq$ $=$

    $>$ $\geq$ $\neq$

11. Find: $9 + \frac{45}{-9}$

12. Find: $8 + 2[3 - 4(7 - 2)]$

Integer Exponents

13. Identify the base and the exponent, then calculate: 4^3

14. Simplify using the properties of exponent: $4^{13} \cdot 4^6$

15. Find: 23^0

16. Write using an exponent:
    $7 \cdot 7 \cdot 7 \cdot 7 \cdot 7 \cdot 7 \cdot 7 \cdot 7 \cdot 7 \cdot 7 \cdot 7 \cdot 7$

17. Simplify using the properties of exponents: $\frac{7^{13} \cdot 12^6}{7^8 \cdot 12^2}$

18. Simplify: $2^3 + 2^5$

19. Write using an exponent: $y \cdot y \cdot y \cdot y \cdot y \cdot y$

20. Simplify using the properties of exponents: $\left(x^3 \cdot y^6\right)^4$

21. Simplify : $\left(2^3 \cdot 3^2\right)$

22. Write using exponents: $3 \cdot 3 \cdot y \cdot y \cdot y \cdot y \cdot y \cdot z \cdot z \cdot z \cdot z$

23. Simplify using the properties of exponents: $\left(\frac{1}{2}x\right)^3 \cdot x^2 \cdot (2y)^3$

24. Simplify using the properties of exponents: $\left[\frac{(2x)^3}{(2y)^2}\right]^3 \cdot y^6 \cdot x^4$

APPLY

Practice Problems

Here are some additional practice problems for you to try.

Real Numbers and Notation

1. Circle the true statements.

   $7 = 12$ $\quad 3 \geq 0$ $\quad 2 \neq 8$

   $2 \leq 2$ $\quad 15 < 2$

2. Circle the true statements.

   $4 \neq 6$ $\quad 7 \leq 7$ $\quad 2 > 5$

   $6 \geq 0$ $\quad 13 < 1$

3. Find the absolute values:

   a. $|3.5|$ c. $|-0.005|$ e. $|36.1|$

   b. $|36|$ d. $|-14.33|$

4. Find the absolute values:

   a. $|0.26|$ c. $|-0.13|$ e. $|-2.69|$

   b. $|-28|$ d. $|15.22|$

5. Find using the Distributive Property: $6 \cdot (11 + 5)$
6. Find using the Distributive Property: $(10 + 5) \cdot 7$
7. Find using the Distributive Property: $7 \cdot (9 - 3)$
8. Which of the following numbers are not integers?

   $-4, 5.2, 7, -\sqrt{8}, \frac{7}{8}, \sqrt{4}$

9. Which of the following numbers are not integers?

   $-2, \sqrt{7}, 4, -\frac{1}{3}, 0, -\sqrt{36}$

10. Which of the following numbers are integers?

    $-21, \sqrt{100}, 3.75, \frac{5}{1}, \frac{1}{5}, \sqrt{24}$

11. Find the additive inverse of 8 and find the multiplicative inverse of 8.
12. Find the additive inverse of 5 and find the multiplicative inverse of 5.
13. Find the additive inverse of $\frac{2}{3}$ and find the multiplicative inverse of $\frac{2}{3}$.
14. Which of the following is not a rational number?

    $\pi, -65, 2.257, \sqrt{7}, -3$

15. Which of the following is not a rational number?

    $-1, 0, 0.14, \sqrt{19}$

16. Which of the following is a rational number?

    $4, \sqrt{2}, \frac{3}{5}, -\sqrt{5}$

17. Find: $8 - (-3) - 10$
18. Find: $9 - 17 - (-26)$
19. Find: $6 - (-2) - 7$
20. Find: $10 - \left(3 - \frac{16}{4}\right)$
21. Find: $7 + \left(\frac{9}{3} - 5\right)$
22. Find: $6 + \left(2 - \frac{27}{3}\right)$
23. Find: $-33 - (-5) - (-27)$
24. Find: $16 - (-4) \cdot (-3)$
25. Find: $-15 - (-4) - (-9)$
26. Find: $-\sqrt{45} + \sqrt{45}$
27. Find: $(\sqrt{17}) + -(\sqrt{17})$
28. Find: $\sqrt{34} - \sqrt{34}$

Integer Exponents

29. Identify the base and the exponent, then calculate: 3^5

30. Identify the base and the exponent, then calculate: 2^5

31. Find: $(-5)^0$

32. Find: $-(5^0)$

33. Write using exponents: $m \cdot m \cdot m \cdot m \cdot m \cdot m \cdot n \cdot n \cdot n \cdot n$

34. Write using exponents: $x \cdot x \cdot x \cdot x \cdot y \cdot y \cdot y \cdot z$

35. Write using exponents: $a \cdot a \cdot a \cdot b \cdot b \cdot b \cdot b \cdot b \cdot b \cdot c \cdot c$

36. Simplify: $3^2 + 4^3$

37. Simplify: $4^2 + 2^3$

38. Simplify: $3^4 - 2^6$

39. Simplify using the properties of exponents: $7^5 - 7^4 \cdot 7^6$

40. Simplify using the properties of exponents: $8^3 \cdot 8^5 + 8^7$

41. Simplify using the properties of exponents: $\frac{5^{13}}{5^6} + 5^4$

42. Simplify using the properties of exponents: $x^3 \cdot y^2 \cdot z^4 \cdot x$

43. Simplify using the properties of exponents: $m^2 \cdot n^7 \cdot m^4 \cdot n^3$

44. Simplify using the properties of exponents: $a^3 \cdot b^9 \cdot c \cdot b^4$

45. Simplify using the properties of exponents: $(a^4 \cdot b^2)^5$

46. Simplify using the properties of exponents: $(x^3 \cdot y^5)^7$

47. Simplify using the properties of exponents: $(x^5 \cdot y^7 \cdot z^8)^9$

48. Simplify using the properties of exponents: $\frac{a^8 \cdot b^9}{a \cdot b^3}$

49. Simplify using the properties of exponents: $\frac{x^{12} \cdot y^8}{x^4 \cdot y^6}$

50. Simplify using the properties of exponents: $\frac{a^5 \cdot b^{12} \cdot c^7}{a^5 \cdot b^6 \cdot c^5}$

51. Simplify using the properties of exponents: $(2a)^3 \cdot a^5 \cdot (3b)^4$

52. Simplify using the properties of exponents: $\left(\frac{1}{3}x\right)^2 \cdot x^4 \cdot (2y)^5$

53. Simplify using the properties of exponents: $\left(\frac{3a}{b^2}\right)^3 \cdot b^7 \cdot \left(\frac{2}{3}a\right)^2$

54. Simplify using the properties of exponents: $\frac{15a^{10}}{5a^5}$

55. Simplify using the properties of exponents: $\frac{4x^6}{2x^8}$

56. Simplify using the properties of exponents: $\frac{3b^3}{18b^{11}}$