CISM COURSES AND LECTURES
INTERNATIONAL CENTRE FOR MECHANICAL SCIENCES

MECHANICS OF MICROPOLAR MEDIA

Edited by O. Brulin and R.K.T. Hsieh

WORLD SCIENTIFIC

INTERNATIONAL CENTRE FOR MECHANICAL SCIENCES

COURSES AND LECTURES

Mechanics of Micropolar Media

Edited by O. Brulin and R.K.T. Hsieh

World Scientific Singapore
1982

World Scientific Publishing Co Pte Ltd
P.O.Box 128
Farrer Road
Singapore 9128

ISBN 9971-950-02-2
Printed by Singapore National Printers (Pte) Ltd

PREFACE

This book is essentially made up of the notes of the lectures the seven authors gave at the International Centre for Mechanical Sciences in Udine in July 1979.

Material with microstructure has generated a tremendous expansion of interest during the past fifteen years. In particular this concerns the theory of Cosserat media with six degrees of freedom as well as the theory based on the notion of directors in relation to more general micropolar continua. This book attempts to provide an up-to-date and reasonably concise summary of our understanding of micropolar materials. Both asymmetric elasticity and fluids are covered. It is hoped that students with physics, mathematics and mechanics backgrounds as well as professional will find this treatise useful for study and reference.

As shown by the Table of Contents, the unusually broad scope of this work includes the most diverse aspects of such materials. The chapters range from discussion of micropolar molecular models to the analysis of structural models, from linear to nonlinear theories and from electromagnetic, thermal, viscous effects to lattice defects. The subjects are treated from both theoretical and experimental points of view with an emphasis on the physical bases, potentialities and limitations of such media. An effort was made to discuss each topic selectively rather than encyclopedically and to incorporate in most chapters a discussion of the fundamentals. Efforts were also made to establish a uniform notation throughout the exposition and a reasonable degree of coherence among the chapters. For practical reasons, no attempt was made to achieve unity of approach and style.

The editors and the co-authors wish to express their deep appreciation to Professors W. Nowacki, W. Olszak and to the officers of C.I.S.M. for inviting them to give the lectures and for ensuring that their stay in Udine was both enjoyable and rewarding. Thanks are also due to the World Scientific Publishing Co. Pte. Ltd. for friendly cooperation.

O. Brulin and R.K.T. Hsieh
Stockholm, June 1981

CONTENTS

MICROPOLAR PHENOMENA IN ORDERED STRUCTURES

Inga Fischer-Hjalmars
Institute of Theoretical Physics
University of Stockholm
Stockholm
SWEDEN

Mechanics of Micropolar Media
Edited by O. Brulin and R.K.T. Hsieh

Contents

1. A model for coupled rotation-displacement modes

1.1 Introduction

Consider a three-dimensional lattice of particles of finite size. We are going to study the propagation of elastic waves in such a lattice. These waves can travel in many different directions. For general lattices and in general directions of propagation the displacements of the particles will have both longitudinal and transverse components. When the lattice has orthogonal symmetry, e.g. as in Fig. 1, it is possible to find directions of propagation such that the elastic wave can be decoupled into purely transverse and purely longitudinal components, cf. Refs. [1,2].

1.2 Lattice of identical particles

Let us first consider a lattice with one kind of particles. This case has been treated by Askar [3]. The central forces between the particles can be represented by stretchable springs and the non-central forces by flexible beams. When we are considering small displacements the total motion can be obtained by superposition of elementary motions originating from the various kinds of forces.

1.2.1 Contribution from central forces [1]

In the following only such cases will be discussed where the general motion can be decomposed into Fourier components with either pure longitudinal or pure transverse polarization as shown in Figs. 2 and 3. All particles lying in planes perpendicular to the propagation vector $\bar{q}$ are then displaced by the same amount. The force on the

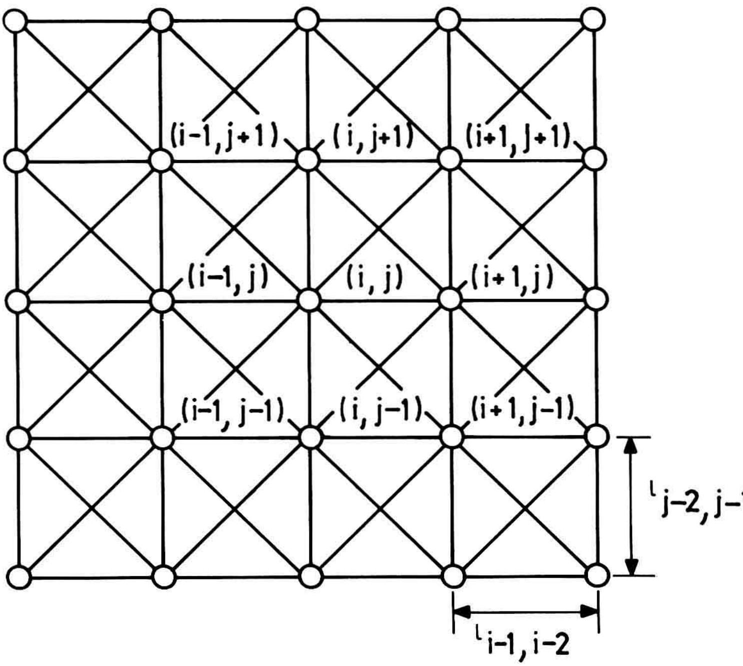

Fig. 1. Example of orthogonal lattice structure.

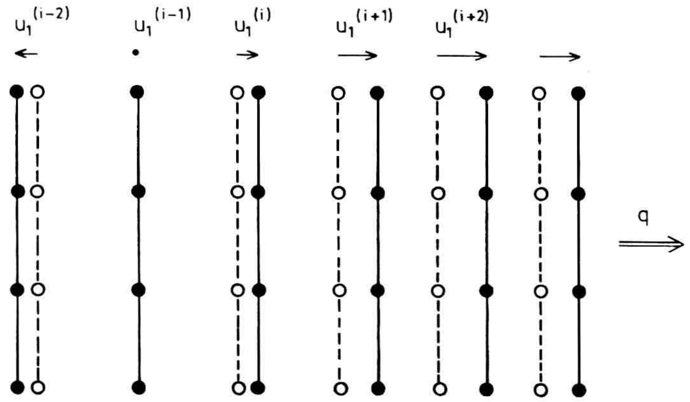

Fig. 2. Longitudinal wave.

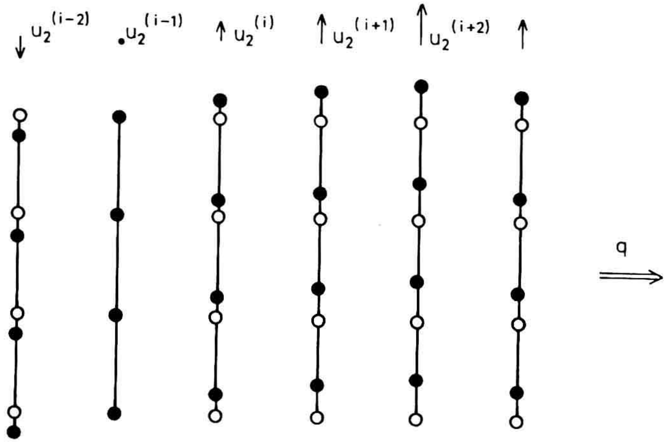

Fig. 3. Transverse wave.

plane labeled i from the plane labeled i+n is proportional to $u_k^{(i+n)} - u_k^{(i)}$, k = 1,2,3. Let the force constants $C^{k,n}$ (defined for one particle of the plane) refer to the interaction between planes at a distance n from each other. The total force is obtained by summation over n. Thus the central forces on the particle i can be written

$$F_k^{(i)} = \sum_n C^{k,n} (u_k^{(i+n)} - u_k^{(i)}) \tag{1.1}$$

When only interaction between neighbouring planes is included the sums over n are reduced to two terms only, viz. n = ±1.

1.2.2 Contribution from non-central forces [2,3]

For representation of non-central interaction between the particles we shall choose extensible and flexible beams. At present let us suppose that the beams are rigidly attached to the particles. Let $\hat{\xi}$ be the direction of the beam. Let the particle-motion be composed of transverse displacements in the $\hat{\eta}$-direction and rotations ψ around axes in the $\hat{\zeta}$-direction, choosing ψ to be positive in the counter-clock-wise direction. The total shear force between two particle planes with the arrangement of Figs. 1-3 will give rise to purely transverse displacements. We shall therefore let a single beam in the x-direction represent the non-central interaction between a particle and all particles in another plane. The shear forces T and moments M, acting on the beam between the particles labeled i and i+n are thus [2]:

$$T_{i,i+n} = K(u_2^{(i)} - u_2^{(i+n)}) + \frac{1}{2} \ell_{i,i+n} K(\psi_3^{(i)} + \psi_3^{(i+n)})$$

$$T_{i+n,i} = K(u_2^{(i)} - u_2^{(i+n)}) + \frac{1}{2} \ell_{i,i+n} K(\psi_3^{(i+n)} + \psi_3^{(i)}) \tag{1.2}$$

$$M_{i,i+n} = \frac{1}{2}\ell_{i,i+n} K(u_2^{(i)} - u_2^{(i+n)}) + \frac{1}{6}\ell_{i,i+n}^2 K(2\psi_3^{(i)} + \psi_3^{(i+n)})$$

$$M_{i+n,i} = \frac{1}{2}\ell_{i,i+n} K(u_2^{(i)} - u_2^{(i+n)} + \frac{1}{6} \ell_{i,i+n}^2 K(2\psi_3^{(i+n)} + \psi_3^{(i)}) \tag{1.3}$$

1.2.3 Motion of the particles

Let the mass of the particle be M and its moment of inertia J. We shall first study the case where all distances between the planes are equal, i.e. $\ell_{i,i+n}$ = na, and only interaction between neighbours is included. Putting $C^1 = C^\ell$, $C^k = C^t$, k = 2,3, the equations of motion of the particle i can be written [3]:

$$M\,\ddot{u}_1^{(i)} = C^\ell(u_1^{(i+1)} + u_1^{(i-1)} - 2u_1^{(i)}) \tag{1.4}$$

$$M\,\ddot{u}_2^{(i)} = (C^t+K)(u_2^{(i+1)} + u_2^{(i-1)} - 2u_2^{(i)}) - \tfrac{1}{2}aK(\psi_3^{(i+1)} - \psi_3^{(i-1)}) \tag{1.5}$$

$$J\,\ddot{\psi}_3^{(i)} = \tfrac{1}{2}aK(u_2^{(i+1)} - u_2^{(i-1)}) - \tfrac{1}{6}a^2K(\psi_3^{(i+1)} + \psi_3^{(i-1)} + 4\psi_3^{(i)}) \tag{1.6}$$

and equations similar to (1.5-6) for u_3 and ψ_2. We seek solutions of the form of waves travelling in the $\hat{x}$-direction with the wave vector $\bar{q}$

$$(u_k,\ \psi_\ell) = (U_k,\ \Psi_\ell)\ \exp[i(qx - \omega t)] \tag{1.7}$$

$$u_k^{(i+n)} = u_k^{(i)} \exp(i\,q\,n\,a)\ ,\ \text{etc.} \tag{1.8}$$

Since eq. (1.4) is not coupled to (1.5-6) we find directly for the longitudinal mode U_1 [1]:

$$\omega_\ell^2 = \frac{2C^\ell}{M}\,(1 - \cos(qa)) = \frac{4C^\ell}{M}\,\sin^2(\tfrac{1}{2}qa) \tag{1.9}$$

The group velocity $v_{gr} = d\omega/dq$ is

$$v_{gr,\ell} = a(C^\ell/M)^{\frac{1}{2}}\,\cos(\tfrac{1}{2}\,qa) \tag{1.10}$$

In the long wavelength limit, i.e. q small, we have

$$(\omega_\ell^0)^2 = (C^\ell/M)\ (qa)^2 \tag{1.11}$$

The phase velocity c_ℓ of the longitudinal wave is in this limit

$$c_\ell^0 = \omega_\ell/(2\pi q) = a/(2\pi)(C^\ell/M)^{\frac{1}{2}} \qquad (1.12)$$

and

$$v_{gr,\ell}^0 = a(C^\ell/M)^{\frac{1}{2}}$$

Since we have a discrete lattice model it is not meaningful to discuss wavelengths shorter than the lattice parameter a.

Hence

$$-\frac{\pi}{a} \leq q \leq \frac{\pi}{a} \qquad (1.13)$$

$$q_{max} = \pm \frac{\pi}{a}$$

As is well known the q-range (1.13) is referred to as the first Brillouin zone. At the zone boundary we have

$$(\omega_\ell^{max})^2 = 4C^\ell/M \qquad (1.14)$$

$$v_{gr,\ell}^{max} = 0$$

The general shape of the dispersion curve is shown in Fig. 4.

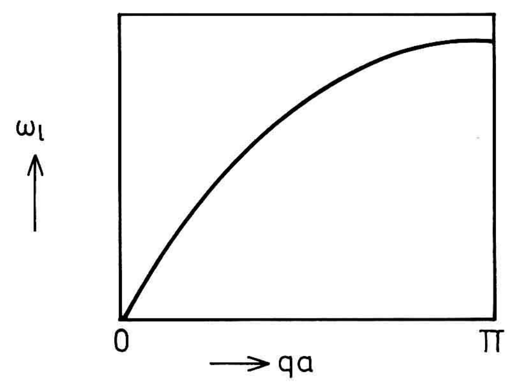

Fig. 4. One-particle lattice.
Dispersion curve of longitudinal mode.

The transverse and rotational modes, U_2 and Ψ_3, are coupled through eqs. (1.5-6). With (1.7-8) we find [3]:

$$[2(C^t + K)(1-\cos(qa)) - M\omega^2] U_2 + iaK \sin(qa) \Psi_3 = 0$$

$$- iaK \sin(qa) U_2 + [\tfrac{1}{3} a^2 K(\cos(qa) + 2) - J\omega^2]\Psi_3 = 0 \qquad (1.15)$$

Although it is possible to solve the secular equation obtained from (1.15) explicitely, we shall not write down these rather imperspicuous expressions. Let us instead look at the limiting cases.

In the long wavelength limit the coupling between the two modes becomes negligible. The displacement u_k will lead to an acoustical mode very similar to the longitudinal mode with solution (1.11-12), although C^ℓ will be replaced by $C^t + K$:

$$(\omega_t^0)^2 = M^{-1}(C^t + K)(qa)^2$$
$$v_{gr,t}^0 = a[(C^t + K)/M]^{\frac{1}{2}} \qquad (1.16)$$

The rotational mode, on the other hand, will be independent of q:

$$(\omega_r^0)^2 = a^2 K/J$$
$$v_{gr,r}^0 = 0 \qquad (1.17)$$

At the opposite limit, $qa = \pi$, the two modes will again be uncoupled. Again, U_2 will be similar to U_1, i.e.

$$(\omega_t^{max})^2 = 4(C^t + K)/M$$
$$v_{gr,t}^{max} = 0 \qquad (1.18)$$

and the rotational mode will also be q-independent

$$(\omega_r^{max})^2 = a^2K/(3J)$$

$$v_{gr,r}^{max} = 0 \qquad (1.19)$$

The motion between these two limits depends upon the relative size of the parameters, C^t, K and Ma^2, J. (1.16-19) shows that ω_t will increase with q and ω_r will decrease. Physically reasonable parameter values may imply that $\omega_t^{max} > \omega_r^{max}$. If so, there will be a regime of q-values where the two modes are strongly coupled. The general picture may be similar to that displayed in Fig. 5.

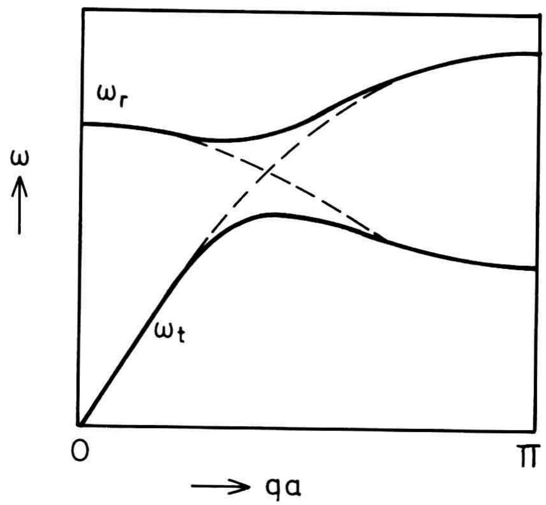

Fig. 5. One-particle lattice. Dispersion curves of transversal and rotational modes.

2. Lattice of two kinds of alternating particles:

longitudinal motion [4,5]

As in case of one-particle lattices there are several symmetries of lattices built from two different kinds of particles that can be treated with the same formalism as a one-dimensional array of particles. Our discussion will be limited to such cases.

2.1 Equations of motion

Let the model array of particles be as shown in Fig. 6.

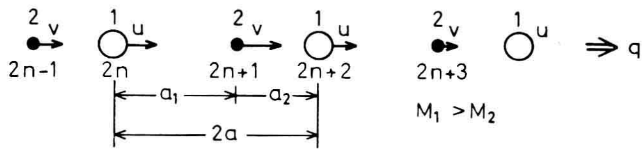

Fig. 6. Longitudinal motion of two-particle lattice.

We shall begin with a study of the longitudinal displacement mode. Consider two kinds of particles with masses M_1 and M_2, $M_1 > M_2$. Let the distances between them be a_1 and a_2, the corresponding force constants C_1 and C_2, and the longitudinal displacements u and v respectively, cf Fig. 6. The equations of motion are

$$M_1\ddot{u}^{(2n)} = -C_2(u^{(2n)} - v^{(2n-1)}) + C_1(v^{(2n+1)} - u^{(2n)})$$

$$M_2\ddot{v}^{(2n+1)} = -C_1(v^{(2n+1)} - u^{(2n)}) + C_2(u^{(2n+2)} - v^{(2n+1)}) \quad (2.1)$$

Assuming travelling waves, cf (1.7), we put

$$u^{(2n+2)} = u^{(2n)} \exp\left[iq(a_1 + a_2)\right]$$

$$v^{(2n+1)} = v^{(2n)} \exp(iqa_1) \text{ etc.} \quad (2.2)$$

Introduction of (2.2) into (2.1) gives

$$(C_1 + C_2 - M_1\omega^2)U-\left[C_1\exp(iqa_1) + C_2\exp(-iqa_2)\right]V = 0$$

$$\left[-C_1\exp(-iqa_1) - C_2\exp(iqa_2)\right]U + (C_1 + C_2 - M_2\omega^2)V = 0 \quad (2.3)$$

To simplify the following analysis we shall assume $a_1 > a_2$ and introduce

$$C_1 = C + \delta$$
$$C_2 = C - \delta \qquad (2.4)$$

$$a_1 = a + \varepsilon$$
$$a_2 = a - \varepsilon \qquad (2.5)$$

(2.3) can now be written

$$(2C - M_1\omega^2)U - [2C \cos(qa) + i\, 2\delta \sin(qa)]e^{iq\varepsilon}\, V = 0$$
$$- [2C \cos(qa) - i\, 2\delta \sin(qa)]U + (2C - M_2\omega^2)e^{iq\varepsilon}\, V = 0 \qquad (2.6)$$

Expansion of the corresponding secular determinant gives

$$M_1M_2\omega^4 - 2C(M_1 + M_2)\omega^2 + 2(C^2 - \delta^2)[1 - \cos(2qa)] = 0 \qquad (2.7)$$

We shall also make use of the total mass M and the reduced mass $\bar{M}$, defined as

$$M = M_1 + M_2 \qquad (2.8)$$

$$\bar{M}^{-1} = M_1^{-1} + M_2^{-1} \qquad (2.9)$$

or $\bar{M} = M_1M_2/M$

2.2 Solution in the long wavelength limit (qa small)

In this limit we expand all functions of qa and discard the terms $O(q^3a^3)$. One solution of (2.7), called the <u>acoustical</u> solution is

$$\omega_{ac}^2 = q^2a^2\,\frac{2C}{M}\,(1 - \delta^2/C^2) + O(q^3a^3) \qquad (2.10)$$

$$v_{gr,ac} \simeq a(2C/M)^{\frac{1}{2}} \qquad (2.11)$$

Comparison between (2.10-11) and (1.11-12) shows that the acoustical solution has the same general appearance as the solution of the one-