The Mathematics of Finance:

Modeling and Hedging

Joseph Stampfli
Victor Goodman

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THE MATHEMATICS OF FINANCE
Modeling and Hedging

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PREFACE

Throughout the nineties, we have seen the synergistic union of mathematics, finance, the computer, and the global economy. Currency markets trade two trillion dollars per day, and sophisticated financial derivatives such as options, swaps, and quantos are commonplace.

Since the appearance of the Black-Scholes formula in 1973, the financial community has embraced an abundant and ever-expanding array of mathematical tools and models. Enrollment in courses presenting these applications of mathematical finance has exploded at schools everywhere. It is driven by the attraction of the material, coupled with enormous employment demand. We expect that the twenty-first century will see even greater growth in these areas, following Kurzweil's law of accelerating returns. The practical analysis of a broad range of market transactions and activities has converted many market devotees to this mode of thinking.

This textbook explains the basic financial and mathematical concepts used in modeling and hedging. Each topic is introduced with the assumption that the reader has had little or no previous exposure to financial matters or to the activities that are common to major equity markets. Exercises and examples illustrate these topics. Often an exercise or example uses real market data.

To the Instructor

A complete, well-balanced course at the undergraduate level can be based on Chapters 2, 3, 5, 6, 7, 8, and 9. An instructor might touch only briefly on Chapter 1 as an introduction to the financial terminology and to strategies that are employed in trading equity shares. You might wish to return to Chapter 1 repeatedly as you progress through the textbook; the chapter is always there as a convenient reference for market transactions and terminology.

Most undergraduate students seem to be very comfortable with computers, and they appear to pick up the ins and outs of software packages such as Maple™, *Mathematica*™, and Microsoft® Excel very quickly. Each instructor will have to evaluate the proficiency of his or her own students in this area. For example, we have found that Excel is readily available on the Indiana University campus and that students are comfortable in preparing data and reports using this software.

Acknowledgments

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In November 1999, Joseph Stampfli presented several lectures on financial mathematics at a workshop on this topic in Bangkok, Thailand, sponsored by Mahidol University. We would like to thank the university and, in particular, Professor Yongwemon Lenbury and Ponchai Matangkasombut, then Dean and now President of the university, for their gracious hospitality throughout the visit. It was a truly memorable experience.

We would also like to thank the editorial and production teams at Brooks/Cole for their continuous and timely help. In particular, Gary Ostedt and Carol Benedict did everything an editorial team can do and more. Several unexpected crises arose as the book progressed, and Gary guided us through them with patience, wisdom, and humor. We would also like to thank the other members of the Brooks/Cole team: Mary Vezilich, Production Coordinator; Karin Sandberg, Marketing Manager; Sue Ewing, Permissions Editor; and Samantha Cabaluna, Marketing Communications. We would also like to thank Kris Engberg of Publication Services, who helped us solve hundreds of problems, both large and small; Jerome Colburn, whose contributions as copy editor turned limp doggerel into sparkling prose; and Jason Brown and his production team.

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A web site for this book is maintained at http://www.indiana.edu/~iubmtc/mathfinance/.

Victor Goodman
Joseph Stampfli

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CHAPTER 1

FINANCIAL MARKETS

If you can look into the seeds of time,
And say which grain will grow and which will not,
Speak then to me...

Shakespeare, *Macbeth,* Act I, Scene ii

Note: This chapter is intended to be a glossary. It is designed to introduce concepts, ideas, and definitions as they are needed. We do not recommend that one work through the entire chapter as a unit. Visit it sparingly as needed.

1.1 MARKETS AND MATH

Nearly everyone has heard of the New York, London, and Tokyo stock exchanges. Reports of the trading activity in these markets frequently make the front page of newspapers and are often featured on evening television newscasts. There are many other financial markets. Each of these has a character determined by the type of financial objects being exchanged.

The most important markets to be discussed in this book are *stock* markets, *bond* markets, *currency* markets, and *futures and options* markets. These financial terms will be explained later. But first we draw your attention to the fact that every item that is exchanged, or **traded,** on some market is of one of two types.

The traded item may be a **basic equity,** such as a stock, a bond, or a unit of currency. Or the item's value may be indirectly *derived from* the value of some other traded equity. If so, its future price is tied to the price of another equity on a future date. In this case, the item is a **financial derivative;** the equity it refers to is termed the **underlying equity.**

This chapter contains many examples of financial derivatives. Each example will be thoroughly explained in order to make the derivative concept clear to the reader. Our examples will be options based on stocks, bonds, and currencies. Also, we will discuss *futures* and options on futures.

Mathematics enters this subject in a serious way when we try to relate a derivative price to the price of the underlying equity. Mathematically based arguments give surprisingly accurate estimates of these values.

> *The main objective of this book is to explain the process of computing derivative prices in terms of underlying equity prices.*

We also wish to provide the reader with the mathematical tools and techniques to carry out this process. Through developing an understanding of this process, you will gain insights into how derivatives are used, and you will comprehend the risks associated with creating or trading these assets. These insights into derivative trading provide extra knowledge of how modern equity markets work.

The mathematics in this book will emphasize two financial concepts that have had a startling impact over the last two decades on the way the financial industry views derivative trading.

We will emphasize investments that **replicate** equities, and we will explore mathematical models of how equities behave in the **absence of arbitrage opportunities.**

The combination of these two concepts furnishes a powerful tool for finding prices. An example is overdue at this point. In the next section, we present an example in which a *replicating investment* and the *lack of arbitrage opportunities* give us a price for a derivative. This example is worth careful reading.

1.2 STOCKS AND THEIR DERIVATIVES

A company that needs to raise money can do so by selling its shares to investors. The company is *owned* by its shareholders. These owners possess **shares** or **equity certificates** and may or may not receive **dividends,** depending on whether the company makes a profit and decides to share this with its owners.

What is the value of the company's stock? Its value reflects the views or predictions of investors about the likely dividend payments, future earnings, and resources that the company will control. These uncertainties are resolved (each trading day) by buyers and sellers of the stock. They exercise their views by trading shares in **auction markets** such as the New York, London, and Tokyo stock exchanges. That is, most of the time a stock's value is judged by what someone else is willing to pay for it on a given day.

What is a stock derivative? It is a specific contract whose value at some future date will depend *entirely* on the stock's future values. The person or firm who formulates this contract and offers it for sale is termed the **writer.** The person or firm who purchases the contract is termed the **holder.** The stock that the contract is based on is termed the **underlying equity.**

What is a derivative worth? The terms of such a contract are crucial in any estimation of its value. As our first example, we choose a derivative with a simple structure so that our main financial concepts, *replicating an equity* and *lack of arbitrage opportunities*, can be easily explained. These will give us a price for this derivative.

We will also explain several trading opportunities in this example that are important for understanding concepts you will encounter later on.

1.2.1 Forward Stock Contracts

It is sometimes convenient to have the assurance that, on some specific future date, one will buy a share of stock for a guaranteed price. This *obligation* to buy in the future is known as a

FORWARD CONTRACT

Here are the contract conditions:

- On a specific date, termed the **expiration date,** the holder of this contract **must** pay a prescribed amount of money, the **exercise price,** to the writer of the contract.
- The writer of the contract **must** deliver one share of stock to the holder on the expiration date.

Figure 1.1 is a pictorial view of the exchange of stock and cash in a forward contract, often called a **forward.**

This contract can either be a good deal for the holder on the day of delivery or be a bad one. The outcome depends on the stock price on the expiration date.

Profit or Loss at Expiration

To state things quantitatively, we will denote the price on that date by S_T and the required exercise price by X. The exercise price, X, is a known quantity. It is also called the **strike price,** and often the expiration date is referred to as the **strike date.**

The profit or loss to the holder at time T is expressed as

$$S_T - X$$

Is there some way to find a profit or loss price formula that will be useful before the contract expires? The question of what the contract should be worth is not an

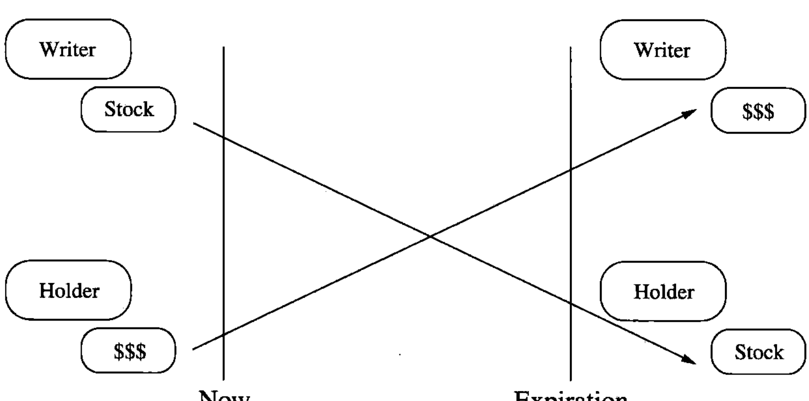

FIGURE 1.1
Forward contract

academic one. Modern markets allow a contract holder to sell the contract in the market or purchase new ones during any trading day. In other words, these instruments are **traded.**

You might imagine that the contract has a high value if today's price is much higher than the exercise price, X, and expiration is not too far away. On the other hand, if today's stock price is quite low, then perhaps it is nearly worthless. We obtain information about its price by creating an investment that *replicates* some other equity.

Replicating Investment

Form an investment choice, called a **portfolio,** that consists of one contract (worth $\$f$) and the following amount of cash:

$$Xe^{-r(T-t)}$$

Now the net worth is

$$f + Xe^{-r(T-t)} \tag{1.1}$$

The exponential factor counteracts interest income.

Any **cash amount** in a portfolio grows by a factor of $e^{r(T-t)}$ in the time from now to expiration. This is reasonable, since the cash will be invested safely. Here, "safely" means that it is placed where the capital will not be at risk from market price changes and can be extracted immediately if needed for some better investment choice. The r denotes the *current* interest rate return on such an investment. During the spring of 2000, an r value of 0.055 per year was used for interest on short-term monetary investments.

The cash amount is adjusted to produce a desired target value for this portfolio on the expiration date of the forward contract.

On that date the portfolio gains a share of stock and pays the exercise price, but the contrived cash amount has grown into exactly the exercise price. In effect, the cash part of the investment disappears and there is no fee.

We can say that on the expiration date this portfolio *replicates* a share of stock. Certainly, the price is correct, since

$$\text{Contract value} + \text{cash amount} = \text{one share of stock}$$

Trading a Portfolio

Now, here is a surprising aspect of modern markets. Their structure allows this portfolio to be traded *before the expiration date* as though it were an equity. In fact, one can buy this type of contract at any time and set aside some cash; this amounts to *purchasing* one unit of the portfolio.

On the other hand, one usually can sell this contract in a market *even if one does not own it.* An investor serves as the writer of a contract and has the same obligations as a writer. When one sells a basic equity such as a stock without *owning it first,* and then purchases the stock later to make delivery, this activity is referred to as **shorting** or **short selling** the equity. It is possible to sell almost any stock short. Clearly, one can "short" the amount of *cash* in the portfolio just by borrowing some money at the short-term interest rate r. In effect, we can *sell* one unit of this portfolio, even if we do not own it to begin with.

We have just explained that one can buy and sell an instrument, the portfolio, that replicates a share of stock on a future date. This leads us to a comparison of prices on an earlier date. We will apply a second major financial principle, *the absence of arbitrage opportunities,* to equate some prices.

First Arbitrage Opportunity

Suppose that today's price is not consistent with its future value. In fact, let us look at the case when

$$\text{Contract value} + \text{cash amount} < \text{one share of stock}$$

This creates a gold mine. The investor can **sell short** quite large amounts of the stock today. This produces instant cash for any investor who is bold enough to sell something that he or she does not own. An investor could use some of the cash to form the correct number of portfolio units to **cover** the short selling.

That is, when the expiration date arrives, the investor neutralizes all the short sales of stock using the **replicating portfolio** value as a stock value. You can see that he or she pockets some cash at the beginning, *regardless of future market behavior,* since it was cheap to cover the short selling.

Second Arbitrage Opportunity

If the reverse situation holds, that is,

$$\text{Contract value} + \text{cash amount} > \text{stock price}$$

the investor could *sell* units of the portfolio *short,* as we explained when we discussed trading the portfolio. Similar arithmetic shows that an investor who covers these short sales with *cheap* stock, purchased immediately after selling short, will still be able to sleep at night.

An investor will not become nervous as the expiration date approaches, because he or she knows that each *short* unit of portfolio will serve only as the liability for exactly one *short* share of stock on this date. Again, some cash is earned at the beginning, *regardless of future market behavior.*

Real markets would not allow either of these money-making schemes to work. We will discuss the reason for this momentarily, but for now, consider the consequence of not having either of the two price inequities discussed above. We obtain the following

No-Arbitrage Price Equation

$$\text{Today's contract value} + \text{cash amount} = \text{today's stock price}$$

We may substitute the values from the net worth equation (1.1) to obtain the formula

$$f + Xe^{-r(T-t)} = S_t$$

The relation can be restated as

$$f = S_t - Xe^{-r(T-t)} \tag{1.2}$$

Equation (1.2) shows that we have "solved" for f. To obtain today's price for this contract, one obtains quotes of today's stock price and the short-term interest rate. Once these ingredients are substituted into the formula above, one has a price for a forward stock contract beginning today.

The price should be recomputed each day as the stock price changes and the time to expiration decreases. The r value is unlikely to change in practice, because forward contracts usually expire within 90 days of their initiation. Since this time span is so short, the return on cash invested is far more sensitive to the time to expiration than it is to the rate for one-month or three-month cash investments.

Example Suppose we have a forward for Eli Lilly stock that will expire 40 days from now. If the exercise price is \$65, and if today's stock price is $\$64\frac{3}{4}$, what is the contract price today?

We will use an r value of 0.055 per year. The quantities we substitute into equation (1.2) are

$$T - t = 40/365 = 0.1096 \text{ (so that } e^{-r(T-t)} = 0.994)$$

and the two quotes, which are

$$S_t = 64.75 \qquad \text{and} \qquad X = 65$$

The end result is

$$f = 64.75 - 65(0.994) = 64.75 - 64.61 = \$0.14$$

Another insight this formula gives is that, for the same strike and stock price in the example, a longer-term contract (say, for six months) will have a larger price, because the $e^{-r(T-t)}$ term changes to 0.974. This illustrates the usefulness of equation (1.2). It allows us to compare prices of forward contracts for various expiration dates and various strike prices.

Why Do Arguments Based on *Replication* and *No Arbitrage* Work?

The price formula given by equation (1.2) and the market price of a forward contract cannot be substantially different. Any sizable difference would induce people to follow one of the two investment tricks we discussed. The *guarantee* of profit would induce them to invest enormous sums of money in one of these schemes. Their activities would, in turn, move prices until the *arbitrage opportunity* was driven out of existence by changes in the underlying stock price. For example, if a large amount of a stock is sold short, its present value goes down because so much of it is offered for sale.

Put another way, the market pressures generated by the investment schemes of people who are *certain* to profit would force stock and contract prices into equilibrium values, where the arbitrage opportunity is missing.