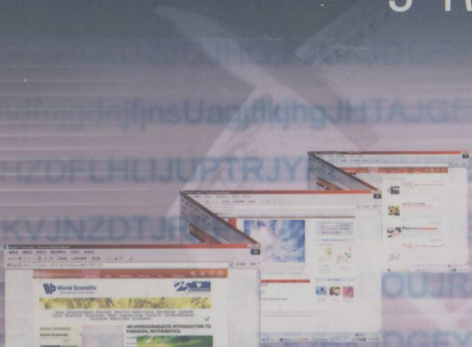
An Undergraduate Introduction to Financial Mathematics

J Robert Buchanan



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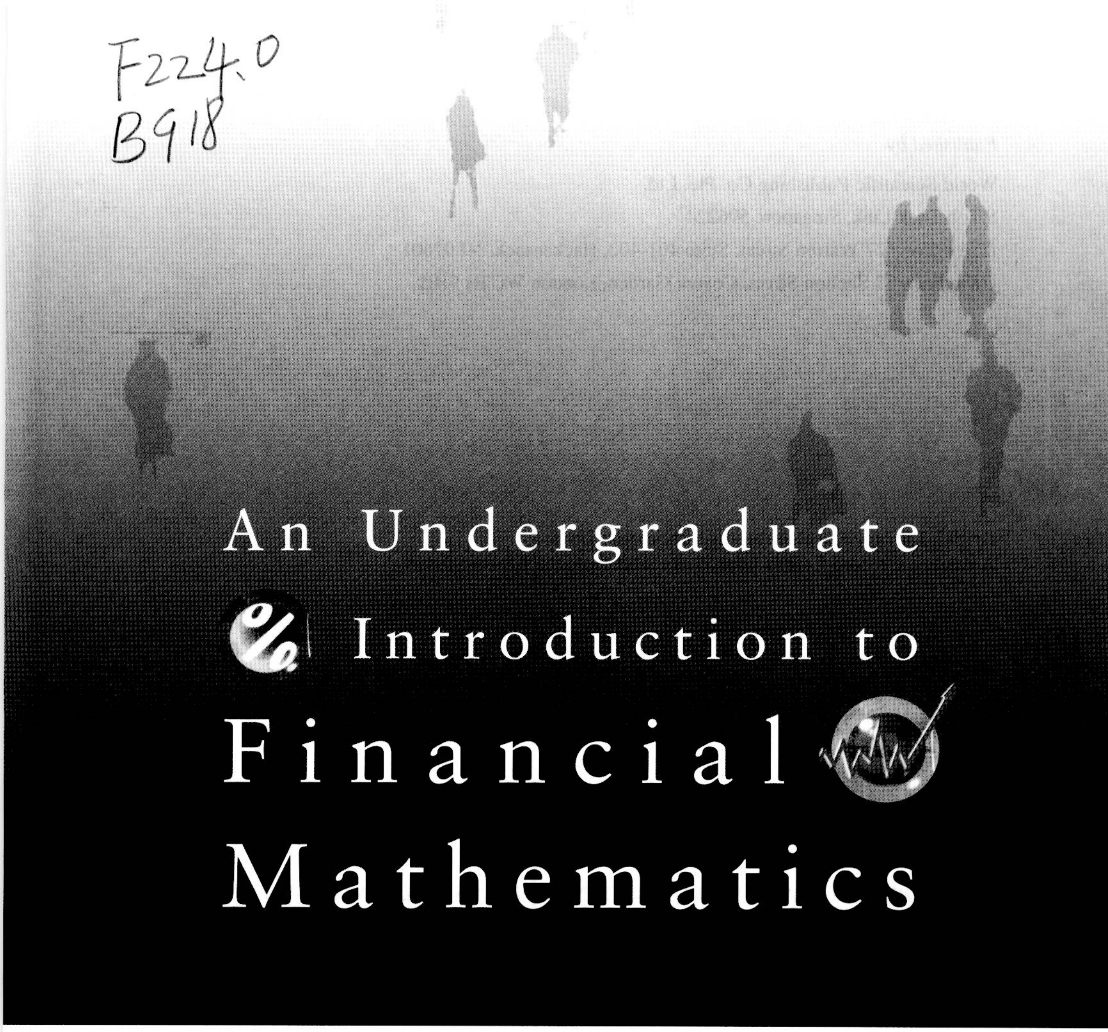

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An Undergraduate Introduction to Financial Mathematics

Dedication

For my wife, Monika.

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Preface

This book is intended for an audience with an undergraduate level of exposure to calculus through elementary multivariable calculus. The book assumes no background on the part of the reader in probability or statistics. One of my objectives in writing this book was to create a readable, reasonably self-contained introduction to financial mathematics for people wanting to learn some of the basics of option pricing and hedging. My desire to write such a book grew out of the need to find an accessible book for undergraduate mathematics majors on the topic of financial mathematics. I have taught such a course now three times and this book grew out of my lecture notes and reading for the course. New titles in financial mathematics appear constantly, so in the time it took me to compose this book there may have appeared several superior works on the subject. Knowing the amount of work required to produce this book, I stand in awe of authors such as those.

This book consists of ten chapters which are intended to be read in order, though the well-prepared reader may be able to skip the first several with no loss of understanding in what comes later. The first chapter is on interest and its role in finance. Both discretely compounded and continuously compounded interest are treated there. The book begins with the theory of interest because this topic is unlikely to scare off any reader no matter how long it has been since they have done any formal mathematics.

The second and third chapters provide an introduction to the concepts of probability and statistics which will be used throughout the remainder of the book. Chapter Two deals with discrete random variables and emphasizes the use of the binomial random variable. Chapter Three introduces continuous random variables and emphasizes the similarities and differences between discrete and continuous random variables. The normal random variable and the close related lognormal random variable are introduced and explored in the latter chapter.

In the fourth chapter the concept of arbitrage is introduced. For readers already well versed in calculus, probability, and statistics, this is the first material which may be unfamiliar to them. The assumption that financial calculations are carried out in an "arbitrage free" setting pervades the remainder of the book. The lack of arbitrage opportunities in financial transactions ensures that it is not possible to make a risk free profit. This chapter includes a discussion of the result from linear algebra and operations research known as the Duality Theorem of Linear Programming.

The fifth chapter introduces the reader to the concepts of random walks and Brownian motion. The random walk underlies the mathematical model of the value of securities such as stocks and other financial instruments whose values are derived from securities. The choice of material to present and the method of presentation is difficult in this chapter due to the complexities and subtleties of stochastic processes. I have attempted to introduce stochastic processes in an intuitive manner and by connecting elementary stochastic models of some processes to their corresponding deterministic counterparts. Itô's Lemma is introduced and an elementary proof of this result is given based on the multivariable form of Taylor's Theorem. Readers whose interest is piqued by material in Chapter Five should consult the bibliography for references to more comprehensive and detailed discussions of stochastic calculus.

Chapter Six introduces the topic of options. Both European and American style options are discussed though the emphasis is on European options. Properties of options such as the Put/Call Parity formula are presented and justified. In this chapter we also derive the partial differential equation and boundary conditions used to price European call and put options. This derivation makes use of the earlier material on arbitrage, stochastic processes and the Put/Call Parity formula.

The seventh chapter develops the solution to the Black-Scholes PDE. There are several different methods commonly used to derive the solution to the PDE and students benefit from different aspects of each derivation. The method I choose to solve the PDE involves the use of the Fourier Transform. Thus this chapter begins with a brief discussion of the Fourier and Inverse Fourier Transforms and their properties. Most three- or four-semester elementary calculus courses include at least an optional section on the Fourier Transform, thus students will have the calculus background necessary to follow this discussion. It also provides exposure to the Fourier

Transform for students who will be later taking a course in PDEs and more importantly exposure for students who will not take such a course. After completing this derivation of the Black-Scholes option pricing formula students should also seek out other derivations in the literature for the purposes of comparison.

Chapter Eight introduces some of the commonly discussed partial derivatives of the Black-Scholes option pricing formula. These partial derivatives help the reader to understand the sensitivity of option prices to movements in the underlying security's value, the risk-free interest rate, and the volatility of the underlying security's value. The collection of partial derivatives introduced in this chapter is commonly referred to as "the Greeks" by many financial practitioners. The Greeks are used in the ninth chapter on hedging strategies for portfolios. Hedging strategies are used to protect the value of a portfolio against movements in the underlying security's value, the risk-free interest rate, and the volatility of the underlying security's value. Mathematically the hedging strategies remove some of the low order terms from the Black-Scholes option pricing formula making it less sensitive to changes in the variables upon which it depends. Chapter Nine will discuss and illustrate several examples of hedging strategies.

Chapter Ten extends the ideas introduced in Chapter Nine by modeling the effects of correlated movements in the values of investments. The tenth chapter discusses several different notions of optimality in selecting portfolios of investments. Some of the classical models of portfolio selection are introduced in this chapter including the Capital Assets Pricing Model (CAPM) and the Minimum Variance Portfolio.

It is the author's hope that students will find this book a useful introduction to financial mathematics and a springboard to further study in this area. Writing this book has been hard, but intellectually rewarding work.

During the summer of 2005 a draft version of this manuscript was used by the author to teach a course in financial mathematics. The author is indebted to the students of that class for finding numerous typographical errors in that earlier version which were corrected before the camera ready copy was sent to the publisher. The author wishes to thank Jill Bachstadt, Jason Buck, Mark Elicker, Kelly Flynn, Jennifer Gomulka, Nicole Hundley, Alicia Kasif, Stephen Kluth, Patrick McDevitt, Jessica Paxton, Christopher Rachor, Timothy Refi, Pamela Wentz, Joshua Wise, and Michael Zrncic.

A list of errata and other information related to this book can be found at a web site I created:

http://banach.millersville.edu/~bob/book/

Please feel free to share your comments, criticism, and (I hope) praise for this work through the email address that can be found at that site.

J. Robert Buchanan
Lancaster, PA, USA
October 31, 2005

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Chapter 1

The Theory of Interest

One of the first types of investments that people learn about is some variation on the savings account. In exchange for the temporary use of an investor's money, a bank or other financial institution agrees to pay **interest**, a percentage of the amount invested, to the investor. There are many different schemes for paying interest. In this chapter we will describe some of the most common types of interest and contrast their differences. Along the way the reader will have the opportunity to renew their acquaintanceship with exponential functions and the geometric series. Since an amount of capital can be invested and earn interest and thus numerically increase in value in the future, the concept of **present value** will be introduced. Present value provides a way of comparing values of investments made at different times in the past, present, and future. As an application of present value, several examples of saving for retirement and calculation of mortgages will be presented. Sometimes investments pay the investor varying amounts of money which change over time. The concept of **rate of return** can be used to convert these payments in effective interest rates, making comparison of investments easier.

1.1 Simple Interest

In exchange for the use of a depositor's money, banks pay a fraction of the account balance back to the depositor. This fractional payment is known as **interest**. The money a bank uses to pay interest is generated by investments and loans that the bank makes with the depositor's money. Interest is paid in many cases at specified times of the year, but nearly always the fraction of the deposited amount used to calculate the interest is called the **interest rate** and is expressed as a percentage paid per year.

For example a credit union may pay 6% annually on savings accounts. This means that if a savings account contains \$100 now, then exactly one year from now the bank will pay the depositor \$6 (which is 6% of \$100) provided the depositor maintains an account balance of \$100 for the entire year.

In this chapter and those that follow, interest rates will be denoted symbolically by r. To simplify the formulas and mathematical calculations, when r is used it will be converted to decimal form even though it may still be referred to as a percentage. The 6% annual interest rate mentioned above would be treated mathematically as $r = 0.06$ per year. The initially deposited amount which earns the interest will be called the **principal amount** and will be denoted P. The sum of the principal amount and any earned interest will be called the **compound amount** and A will represent it symbolically. Therefore the relationship between P, r, and A for a single year period is

$$A = P + Pr = P(1 + r).$$

The interest, once paid to the depositor, is the depositor's to keep. Banks and other financial institutions "pay" the depositor by adding the interest to the depositor's account. Unless the depositor withdraws the interest or some part of the principal, the process begins again for another interest period. Thus two interest periods (think of them as years) after the initial deposit the compound amount would be

$$A = P(1 + r) + P(1 + r)r = P(1 + r)^2.$$

Continuing in this way we can see that t years after the initial deposit of an amount P, the compound amount A will grow to

$$A = P(1 + r)^t \tag{1.1}$$

This is known as the **simple interest** formula. A mathematical "purist" may wish to establish Eq. (1.1) using the principle of induction.

Banks and other interest-paying financial institutions often pay interest more than a single time per year. The simple interest formula must be modified to track the compound amount for interest periods of other than one year.

1.2 Compound Interest

The typical interest bearing savings or checking account will be described an investor as earning a specific annual interest rate *compounded monthly*. In this section will be compare and contrast compound interest to the simple interest case of the previous section. Whenever interest is allowed to earn interest itself, an investment is said to earn **compound interest.** In this situation, part of the interest is paid to the depositor more than once per year. Once paid, the interest begins earning interest. We will let the number of compounding periods per year be n. For example for interest "compounded monthly" $n = 12$. Only two small modifications to the simple interest formula (1.1) are needed to calculate the compound interest. First, it is now necessary to think of the interest rate per compounding period. If the annual interest rate is r, then the interest rate per compounding period is r/n. Second, the elapsed time should be thought of as some number of compounding periods rather than years. Thus with n compounding periods per year, the number of compounding periods in t years is nt. Therefore the formula for compound interest is

$$A = P\left(1 + \frac{r}{n}\right)^{nt}. \tag{1.2}$$

Example 1.1 Suppose an account earns 5.75% annually compounded monthly. If the principal amount is \$3104 then after three and one half years the compound amount will be

$$A = 3104\left(1 + \frac{0.0575}{12}\right)^{(12)(3.5)} = 3794.15.$$

The reader should verify that if the principal in the previous example earned a *simple* interest rate of 5.75% then the compound amount after 3.5 years would be only \$3774.88. Thus happily for the depositor, compound interest builds wealth faster than simple interest. Frequently it is useful to compare an annual interest rate with compounding to an equivalent simple interest, *i.e.* to the simple annual interest rate which would generate the sample amount of interest as the annual compound rate. This equivalent interest rate is called the **effective interest rate.** For the amounts and rates mentioned in the previous example we can find the effective interest rate by solving the equation

$$\begin{aligned} 3104\left(1+\frac{0.0575}{12}\right)^{12} &= 3104(1+r_e) \\ 1.05904 &= 1+r_e \\ 0.05904 &= r_e \end{aligned}$$

Thus the annual interest rate of 5.75% compounded monthly is equivalent to an effective annual simple rate of 5.904%.

Intuitively it seems that more compounding periods per year implies a higher effective annual interest rate. In the next section we will explore the limiting case of frequent compounding going beyond semiannually, quarterly, monthly, weekly, daily, hourly, *etc.* to continuously.

1.3 Continuously Compounded Interest

Mathematically when considering the effect on the compound amount of more frequent compounding, we are contemplating a limiting process. In symbolic form we would like to find the compound amount A which satisfies the equation

$$A = \lim_{n\to\infty} P\left(1+\frac{r}{n}\right)^{nt}. \tag{1.3}$$

Fortunately there is a simple expression for the value of the limit on the right-hand side of Eq. (1.3). We will find it by working on the limit

$$\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^{n}.$$

This limit is indeterminate of the form 1^∞. We will evaluate it through a standard approach using the natural logarithm and l'Hôpital's Rule. The reader should consult an elementary calculus book such as [Smith and Minton (2002)] for more details. We see that if $y = (1+r/n)^n$, then

$$\begin{aligned} \ln y &= \ln\left(1+\frac{r}{n}\right)^{n} \\ &= n\ln(1+r/n) \\ &= \frac{\ln(1+r/n)}{1/n} \end{aligned}$$

which is indeterminate of the form $0/0$ as $n \to \infty$. To apply l'Hôpital's Rule we take the limit of the derivative of the numerator over the derivative of the denominator. Thus

$$\lim_{n\to\infty} \ln y = \lim_{n\to\infty} \frac{\frac{d}{dn}\left(\ln(1+r/n)\right)}{\frac{d}{dn}\left(1/n\right)}$$
$$= \lim_{n\to\infty} \frac{r}{1+r/n}$$
$$= r$$

Thus $\lim_{n\to\infty} y = e^r$. Finally we arrive at the formula for **continuously compounded interest**,

$$A = Pe^{rt}. \tag{1.4}$$

This formula may seem familiar since it is often presented as the exponential growth formula in elementary algebra, precalculus, or calculus. The quantity A has the property that A changes with time t at a rate proportional to A itself.

Example 1.2 Suppose \$3585 is deposited in an account which pays interest at an annual rate of 6.15% compounded continuously. After two and one half years the principal plus earned interest will have grown to

$$A = 3585e^{(0.0615)(2.5)} = 4180.82.$$

The effective simple interest rate is the solution to the equation

$$e^{0.0615} = 1 + r_e$$

which implies $r_e = 6.34305\%$.

1.4 Present Value

One of the themes we will see many times in the study of financial mathematics is the comparison of the value of a particular investment at the present time with the value of the investment at some point in the future. This is the comparison between the **present value** of an investment versus its **future value**. We will see in this section that present and future value play central roles in planning for retirement and determining loan payments. Later in this book present and future values will help us determine a fair price for stock market derivatives.

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