FIFTH EDITION

ANALYTIC GEOMETRY

GORDON FULLER

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PREFACE

This fifth edition of *Analytic Geometry*, like the earlier editions, has been tailored for a first course in the study of the subject. It emphasizes, especially, the basic concepts that are needed in calculus and in many other areas of mathematics.

Users of the earlier editions of the book will observe that every chapter in this edition has been considerably strengthened. This improvement has been achieved by the use of improved exposition, or by the introduction of new material, or by both. Some of the more important changes in this book are as follows:

1. A chapter on transcendental function is now included.
2. Each chapter now ends with a review exercise.
3. Most of the chapters now contain several interesting applied problems.
4. The intersection of lines, along with an exercise pertaining to this subject, has been introduced.
5. The solutions of some chosen trigonometric equations are reviewed. This discussion smooths the way for solving systems of polar coordinate equations.
6. The answers to problems whose numbers are not multiples of three are included in the text, and the answers to problems whose numbers are multiples of three are contained in a booklet, which is available to teachers without charge. This is a very important change, in contrast with the former plan of having the answers to problems divided evenly between the text and the booklet.
7. Almost all the exercises have more problems that they did formerly.

The study of conics and polar coordinates has been deservingly treated more fully than is essential for the study of elementary calculus. If desired, however, the time devoted to this part may be shortened by assigning a limited number of problems from the exercises.

In Chapters 9 and 10 the elements of solid analytic geometry are treated. The first of these chapters deals with quadric surfaces, and the second deals with planes and lines. This order was chosen because a class that takes only one of the two chapters should preferably study space illustrations of second-degree equations. Vectors are introduced and applied in the discussion of planes and lines. This study has been facilitated, of course, by the use of vectors, and it provides the student with more than a passing encounter with this valuable concept.

The solved problems throughout the book bear a close correlation with the problems found in the exercises. This feature, together with the careful exposition, tends to give the book a self-teaching quality. The exercise sections occur at short intervals, and each contains an abundance of problems. Also, many problems of theoretical implication have been included. These, of course, should afford a definite challenge to some of the more knowledgeable students. The five numerical tables in the Appendix should meet any needs that may arise in solving the problems.

This book is intended as a semester course for senior high school students or for freshmen students in colleges and universities. Students in high school with four or five class meetings a week should cover most, if not all, of the book. However, students in college, where three meetings a week are more likely, will, in some instances, not finish the entire book. If this situation should arise, either in high school or in college, we suggest that Sections 7.4, 7.15, 10.9, and 10.10 be the primary candidates for omission.

I would like to express my appreciation for the helpful guidance provided by the reviewers: Charles Cheney (Indiana State University), Peter Cook (Purdue University), and Derald Walling (Texas Tech University).

Special thanks go to Professors Horace Woodward and Robert Parker, both of Texas Tech University, for their valuable assistance with the manuscript.

Lubbock, Texas
December 1978

G. F.

CONTENTS

1

FUNDAMENTAL CONCEPTS

For several centuries geometry and algebra developed slowly, bit by bit, as distinct mathematical disciplines. In 1637, however, a French mathematician and philosopher, René Descartes, published his *La Géométrie*, which introduced a device for unifying these two branches of mathematics. The basic feature of this new process, now called **analytic geometry**, is the use of a coordinate system. By means of coordinate systems algebraic methods can be applied powerfully in the study of geometry, and perhaps of still greater importance is the advantage accruing to algebra by the graphical representation of algebraic equations. Indeed, Descartes' remarkable contribution paved the way for rapid and far-reaching developments in mathematics, and analytic geometry continues to be of great value in mathematical investigations.

1.1 DIRECTED LINES AND SEGMENTS

A line on which one direction is chosen as positive and the opposite direction as negative is called a **directed line**. A segment of the line, consisting of any two points and the part between, is called a **directed line segment**. In Fig. 1.1, the positive direction is indicated by the arrowhead. The points A and B determine a segment, which we denote by AB or BA. We specify that the distance from A to B, measured in the positive direction, is positive; and the distance from B to A, measured in the negative direction, is negative. These two distances, which we denote by $\overrightarrow{AB}$ and $\overrightarrow{BA}$ are called **directed distances**. If the length of the line segment is 3, then $\overrightarrow{AB} = 3$, and $\overrightarrow{BA} = -3$. Distances, therefore, on a directed line segment satisfy the equation

$$\boxed{\overrightarrow{AB} = -\overrightarrow{BA}}$$

Another concept with respect to distance on the segment AB is that of the **undirected distances** between A and B. The undirected distance is the length of

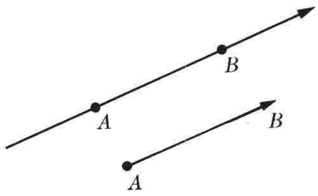

FIGURE 1.1

the segment, which we take as positive. We will use the notation $|AB|$ or $|BA|$ to indicate the positive measurement of distance between A and B, or the length of the line segment AB.

In view of the preceding discussion we may write

$$\overrightarrow{AB} = |AB| = |BA| = 3$$

$$\overrightarrow{BA} = -|AB| = -|BA| = -3$$

Frequently the concept of the absolute value of a number is of particular significance. Relative to this concept, we have the following definition.

Definition 1.1 *The* ***absolute value*** *of a real number a, denoted by $|a|$, is the real number such that*

$$|a| = a \text{ when } a \text{ is positive or zero}$$

$$|a| = -a \text{ when } a \text{ is negative}$$

According to this definition, the absolute value of every nonzero number is positive and the absolute value of zero is zero. Thus,

$$|5| = 5 \qquad |-5| = -(-5) = 5 \qquad |0| = 0$$

Theorem 1.1 *If A, B, and C are three points of a directed line, then the directed distances determined by these points satisfy the equations*

$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC} \qquad \overrightarrow{AC} + \overrightarrow{CB} = \overrightarrow{AB} \qquad \overrightarrow{BA} + \overrightarrow{AC} = \overrightarrow{BC}$$

Proof. If B is between A and C, the distances $\overrightarrow{AB}$, $\overrightarrow{BC}$, and $\overrightarrow{AC}$ all have the same sign, and $\overrightarrow{AC}$ is obviously equal to the sum of the other two (Fig. 1.2). The second and third equations follow readily from the first. To establish the second equation, we add $-\overrightarrow{BC}$ to both sides of the first equation and then use the condition that $-\overrightarrow{BC} = \overrightarrow{CB}$. Thus,

$$\overrightarrow{AB} = \overrightarrow{AC} - \overrightarrow{BC} = \overrightarrow{AC} + \overrightarrow{CB}$$

FIGURE 1.2

1.2 THE REAL NUMBER LINE

A basic concept of analytic geometry is the representation of all real numbers by points on a directed line. The real numbers, we note, consist of the positive numbers, the negative numbers, and zero.

To establish the desired representation, we first choose a direction on a line as positive (to the right in Fig. 1.3) and select a point O of the line, which we call the **origin**, to represent the number zero. Next we mark points at distances 1, 2, 3, and so on, units to the right of the origin. We let the points thus located represent the numbers 1, 2, 3, and so on. In the same way we locate points to the left of the origin to represent the numbers -1, -2, -3, and so on. We now have points assigned to the positive integers, the negative integers, and the integer zero. Numbers whose values are between two consecutive integers have their corresponding points between the points associated with those integers. Thus the number $2\frac{1}{4}$ corresponds to the point $2\frac{1}{4}$ units to the right of the origin. And, in general, any positive number p is represented by the point p units to the right of the origin, and a negative number $-n$ is represented by the point n units to the left of the origin. Further, we assume that every real number corresponds to one point on the line and, conversely, every point on the line corresponds to one real number. This relation of the set of real numbers and the set of points on a directed line is called a **one-to-one correspondence.**

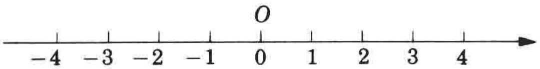

FIGURE 1.3

The directed line of Fig. 1.3, with its points corresponding to real numbers, is called a **real number line.** The number corresponding to a point on the line is called the **coordinate** of the point. Since the positive numbers correspond to points in the chosen positive direction from the origin and the negative numbers correspond to points in the opposite or negative direction from the origin, we shall consider the coordinates of points on a number line to be **directed distances** from the origin. For convenience, we shall sometimes speak of a point as being a number, and vice versa. For example, we may say "the point 5" when we mean "the number 5," and "the number 5" when we mean "the point 5."

1.3 RECTANGULAR COORDINATES

Having obtained a one-to-one correspondence between the points on a line and the system of real numbers, we next develop a scheme for putting the points of a plane into a one-to-one correspondence with a set of ordered pairs of real numbers.

Definition 1.2 *A pair of numbers (x, y) in which x is the first number and y the second number is called an* ***ordered pair.***

We draw a horizontal line and a vertical line meeting at the origin O (Fig. 1.4). The horizontal line OX is called the **x axis** and the vertical line OY, the **y axis.** The x axis and the y axis, taken together, are called the **coordinate axes**, and the plane determined by the coordinate axes is called the **coordinate plane.** The x axis, usually drawn horizontally, is called the **horizontal** axis and the y axis the **vertical** axis. With a convenient unit of length, we make a real number scale on each coordinate axis, letting the origin be the zero point. The positive direction is chosen to the right on the x axis and upward on the y axis, as indicated by the arrowheads in the figure.

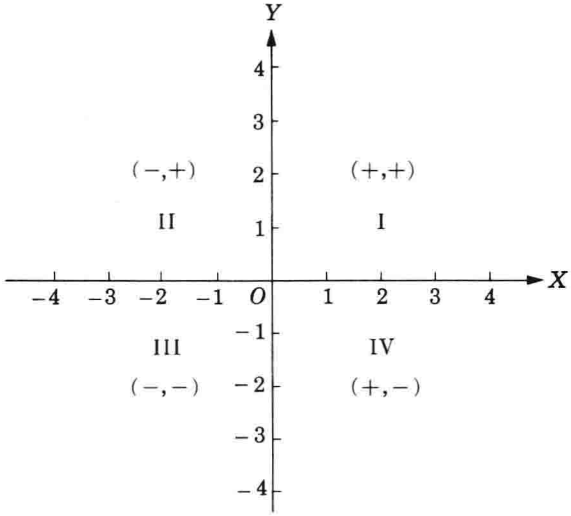

FIGURE 1.4

If P is a point on the coordinate plane, we define the distances of the point from the coordinate axes to be **directed distances.** That is, the distance from the y axis is positive if P is to the right of the y axis and negative if P is to the left, and the distance from the x axis is positive if P is above the x axis and negative if P is below the x axis. Each point P of the plane has associated with it a pair of numbers called **coordinates.** The coordinates are defined in terms of the perpendicular distances from the axes to the point.

Definition 1.3 *The x coordinate, or* ***abscissa****, of a point P is the directed distance from the y axis to the point. The y coordinate, or* ***ordinate****, of a point P is the directed distance from the x axis to the point.*

A point whose abscissa is x and whose ordinate is y is designated by (x, y), in that order, the abscissa always coming first. Hence the coordinates of a point are an ordered pair of numbers. Although a pair of coordinates determines a point, the coordinates themselves are often referred to as a point.

We assume that to any pair of real numbers (coordinates) there corresponds one definite point. Conversely, we assume that to each point of the plane there corresponds one definite pair of coordinates. This relation of points on a plane and pairs of real numbers is called a one-to-one correspondence. The device we have described for obtaining this correspondence is called a **rectangular coordinate system.**

A point of given coordinates is **plotted** by measuring the proper distances from the axes and marking the point thus located. For example, if the coordinates of a point are $(-4, 3)$, the abscissa -4 means the point is 4 units to the left of the y axis and the ordinate 3 (plus sign understood) means the point is 3 units above the x axis. Consequently, we locate the point by going from the origin 4 units to the left along the x axis and then 3 units upward parallel to the y axis (Fig. 1.5).

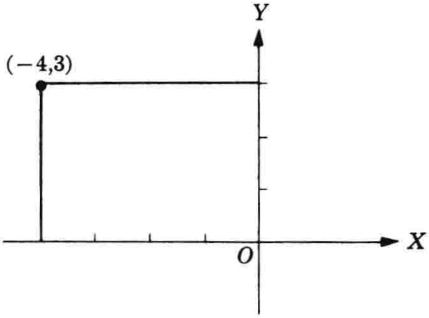

FIGURE 1.5

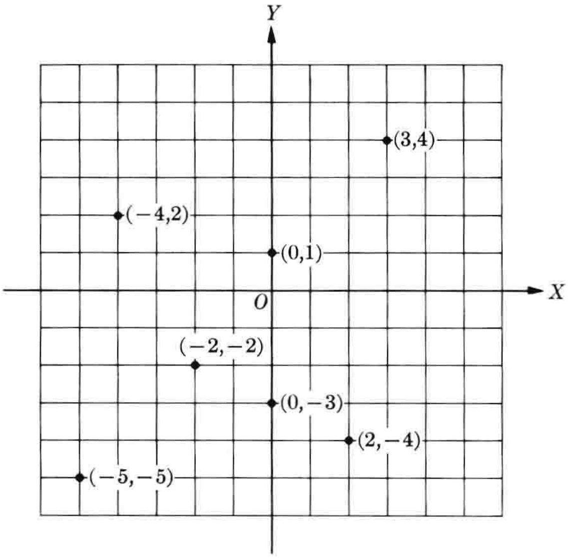

FIGURE 1.6

Similarly, if we wish to plot the point (5, −3), we move 5 units to the right of the origin along the x axis and then 3 units downward (since the ordinate is negative) parallel to the y axis. We have now located the desired point.

Some coordinates and their corresponding points are plotted in Fig. 1.6.

The coordinate axes divide the plane into four parts, called **quadrants**, which are numbered I to IV in Fig. 1.4. The coordinates of a point in the first quadrant are both positive, which is indicated in the figure by (+, +). The signs of the coordinates in each of the other quadrants are similarly indicated.

1.4 DISTANCE BETWEEN TWO POINTS

In many problems the distance between two points of the coordinate plane is required. The distance between any two points, or the length of the line segment connecting them, can be determined from the coordinates of the points. We shall classify a line segment (or line) as **horizontal, vertical**, or **slant**, depending on whether the segment is parallel to the x axis, to the y axis, or to neither axis. In deriving appropriate formulas for the lengths of these kinds of segments, we shall use the idea of directed segments.

Let $P_1(x_1, y)$ and $P_2(x_2, y)$ be two points on a horizontal line, and let A be the point where the line cuts the y axis (Fig. 1.7). We have, by Theorem 1.1,

$$\overrightarrow{AP_1} + \overrightarrow{P_1P_2} = \overrightarrow{AP_2}$$

$$\overrightarrow{P_1P_2} = \overrightarrow{AP_2} - \overrightarrow{AP_1}$$
$$= x_2 - x_1$$

Similarly, for the vertical distance $\overrightarrow{Q_1Q_2}$, we have

$$\overrightarrow{Q_1Q_2} = \overrightarrow{Q_1B} + \overrightarrow{BQ_2}$$
$$= \overrightarrow{BQ_2} - \overrightarrow{BQ_1}$$
$$= y_2 - y_1$$

Hence the directed distance from a first point to a second point on a horizontal

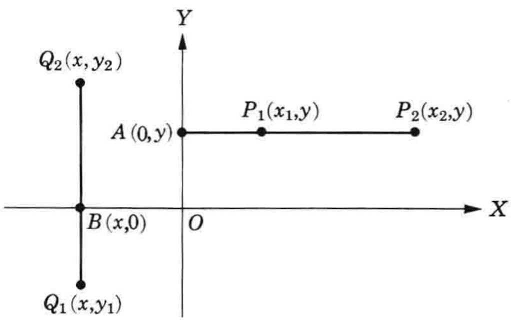

FIGURE 1.7

line is equal to the abscissa of the second point minus the abscissa of the first point. The distance is positive or negative according as the second point is to the right or left of the first point. A corresponding statement can be made relative to a vertical segment.

Inasmuch as the lengths of segments, without regard to direction, are often desired, we state a rule that gives results in positive quantities.

Rule 1 *The length of a horizontal line segment joining two points is the abscissa of the point on the right minus the abscissa of the point on the left.*

The length of a vertical line segment joining two points is the ordinate of the upper point minus the ordinate of the lower point.

We apply these rules to find the lengths of the line segments in Fig. 1.8.

$|AB| = 5 - 1 = 4$ $\qquad$ $|CD| = 6 - (-2) = 6 + 2 = 8$

$|EF| = 1 - (-4) = 1 + 4 = 5$ $\qquad$ $|GH| = -2 - (-5) = -2 + 5 = 3$

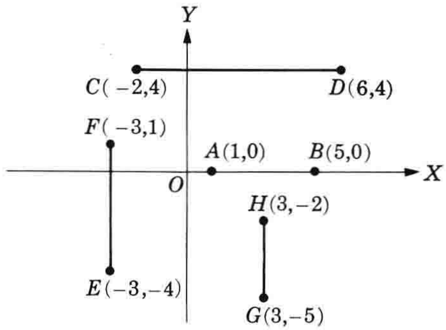

FIGURE 1.8

We next consider the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$, which determine a slant line. Draw a line through P_1 parallel to the x axis and a line through P_2 parallel to the y axis (Fig. 1.9). These two lines intersect at the point R, whose abscissa is x_2

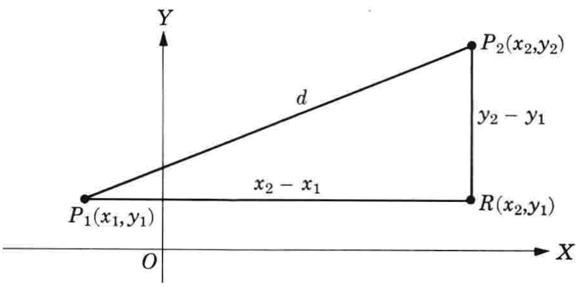

FIGURE 1.9