Manifolds and Geometry

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Brought together in this book are papers from a conference on differential geometry held in Pisa, in honour of one of the world's most highly respected geometers, Eugenio Calabi. The contributions are from many of the leading authorities in this field and together they cover a wide spectrum of topics and give an unsurpassed overview of current research into differential geometry.

Preface

A conference on differential geometry was held in Pisa in September 1993, with the co-operation of the Scuola Normale Superiore di Pisa and the Consiglio Nazionale delle Richerche.

The days of the conference were graced by the active participation of Eugenio Calabi, and we wish to dedicate this volume to him. The conference was organized by a committee consisting of Paolo de Bartolomeis, Franco Tricerri and Edoardo Vesentini.

While this volume was in preparation, Franco Tricerri and his family met with a tragic death in China. We cannot pass without paying a tribute to the memory of a good mathematician and a dear friend.

Paolo de Bartolomeis

Edoardo Vesentini

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Grassman and hyperKähler structures on some spaces of sections of holomorphic bundles

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To Eugenio Calabi

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Abstract

A holomorphic submersion $\pi : Z \to \mathbb{C}P^1$ over a projective line is considered. On an open submanifold M of the complex manifold of sections, a right flat Grassman structure is defined, that is an isomorphism $TM \cong E \otimes H$ of the holomorphic tangent bundle TM onto tensor product of a holomorphic vector bundle E and the trivial bundle $H \cong \mathbb{C}^l$. If a conformal symplectic structure (ω) which depends holomorphically on a fiber is given, then the Grassman structure reduces to a $Sp_{2k} \otimes \mathrm{id}_H$-structure. For $l = 2$ it admits unique torsionless connection and is identified with a complex hyperKähler structure. This construction is applied to the case when a fiber of the submersion π is an adjoint orbit of a complex semisimple Lie group G of automorphisms of π. With the help of the momentum mapping, such submersions are described as being associated with polynomials with coefficients from the Lie algebra of G. The generalized hyperKähler structures on the corresponding space of sections are described.

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Contents

1 Grassman type structures as G-structures

1.1 Notations

We shall assume that all objects considered are complex and holomorphic unless otherwise stated.

In particular M, Q denote complex manifolds, TM is the holomorphic tangent bundle of M, T stands for a complex vector space, $GL(T)$ is the complex general linear group and so on.

1.2 Grassman structure on a vector space

Let T, E, H be vector spaces. A Grassman structure of type (E, H) on a vector space T is an isomorphism $E \otimes H \to T$.

If a Grassman structure is given we will identify T with $E \otimes H$.

Note that the direct product $GL(E) \times GL(H)$ of the general linear groups acts naturally on $T = E \otimes H$ with the kernel $\mathbb{C}^* = \{(a\,\mathrm{id}_E\,, a^{-1}\,\mathrm{id}_H), a \in \mathbb{C}^*\}$. We denote the corresponding subgroup of $GL(T)$ by $GL(E) \otimes GL(H)$. More generally, any subgroups K of $GL(E)$ and L of $GL(H)$ define a subgroup of $GL(E) \otimes GL(H)$ which is denoted by $K \otimes L$.

Two Grassman structures of type (E, H) on a vector space T are called equivalent if they are differ by an element from $GL(E) \otimes GL(H)$.

Lemma 1.1:

1. Let K, L be connected semisimple Lie groups and ρ be an irreducible representation of the group $K \times L$ on a vector space T. Then T admits a Grassman structure $T = E \otimes H$ such that $\rho(K \times L) \subset GL(E) \otimes GL(H)$. This structure is unique up to equivalence.

2. Let ρ be a representation of a connected Lie group L on a vector space T such that L-module T has a decomposition

$$T = rH = H \oplus \ldots \oplus H,$$

where H is an irreducible L-module of dimension > 1. Then there exists a unique (up to an equivalence) Grassman structure $T = E \otimes H, E = \mathbb{C}^r$ on T such that $\rho(L) \in \mathrm{id} \otimes \rho_H(L)$ where ρ_H is the restriction of ρ onto H.

These are standard results from representation theory. Note that (1) is not true for real representations.

1.3 Grassman structures and subordinated structures on a manifold

Let $G \subset GL(T)$ be a (complex) linear group,$T = \mathbb{C}^n$. Recall that a (complex) G-structure on a (complex) manifold M is a principal G-bundle $\pi : Q \to M = Q/G$ over n-dimensional manifold M (with a left holomorphic action of G into Q) together with a displacement form $\theta : TQ \to T$ that is a G-equivariant T-valued strictly horizontal ($\operatorname{Ker}\theta = T^vQ$) 1-form. (Here T^vQ stands for the vertical subbundle of the tangent bundle TQ). This difinition is equivalent to the standard definition of G-structure as a principal G-subbundle of the bundle of coframes. Now we fix a Grassman structure $T = E \otimes H$ in the vector space T.

Definition 1.1: G-structure $\pi : Q \to M$ is called

1. Grassman structure if $G = GL(E) \otimes GL(H)$
2. unimodular Grassman structure if $G = SL(E) \otimes SL(H)$
3. quaternionic Grassman structure if $G = GL(E) \otimes SL_2^{[m]}$, where $SL_2^{[m]}$ is the linear group defined by irreducible representation of SL_2 in the space $H = S^m(\mathbb{C}^2) = \mathbb{C}^{m+1}$
4. bisymplectic structure if $G = Sp(E) \otimes Sp(H)$, where $Sp(E)$, $Sp(H)$ are symplectic groups of the symplectic vector spases E,H
5. symplecto-orthogonal structure if $G = Sp(E) \otimes SO(H)$, where $SO(H)$ is the orthogonal group
6. HyperKähler Grassman structure if $G = Sp(E) \otimes 1$
7. right-flat Grassman structure if $G = GL(E) \otimes 1$

We will refer to these G-structures as G-structures of Grassman type. We note that the Grassman structure was defined and studied firstly by Th. Hangan [Han],see also [Ak],[Man],[Mar],[Bai-East]. Let $\pi : Q \to M$ be a Grassman structure on M. The holomorphic vector bundles $E(\pi), H(\pi)$ with fibers E,H associated with π and the representation of $GL(E) \otimes GL(H)$ in the space E,H are called spinor bundles. By definfition

$$E(\pi) = E \times_G Q = E \times Q/G$$

It is easy to check that

$$TM \simeq E(\pi) \otimes H(\pi)$$

canonically. Conversely, any isomorphism of TM with the tensor product of two holomorphic vector bundles defines a Grassman structure on M.

1.4 D-connections

Let $\pi : Q \to M$ be a G-structure. Note that a horizontal subspace H of the tangent space T_pQ (i.e. a subspace complementary to the vertical subspace T_p^vQ) may be identified with the 1-jet of a section of π. We denote the set of all horizontal subspaces by $J^1(\pi)$. It is a principal bundle over Q with the vector group $\mathrm{Hom}(T, \mathcal{G}) = \mathcal{G} \otimes T^*$ as the structure group where $\mathcal{G}$ is the Lie algebra of G. The structure function of π is defined as a function

$$t : J^1(\pi) \to T \otimes \Lambda^2 T^*$$

given by

$$t_H(u, v) = d\theta(\theta_H^{-1} u, \theta_H^{-1} v)$$

for $H \in J^1(\pi)$, $u, v \in T$, where $\theta_H = \theta\,|_H \colon H \to T$ is the isomorphism defined by the restriction of the displacement form.
A connection in a G-structure π is a G-invariant section of the bundle $J^1(\pi) \to Q$, or, in other words, a G-invariant field

$$H : Q \ni p \mapsto H_p \subset T_pQ,$$

of horizontal subspaces.
The connection form of a connection H is defined as $\mathcal{G}$-valued holomorphic 1-form σ on Q with the kernel H which prolongs the vertical parallelism $\nu : T^vQ \to \mathcal{G}$ (i.e. the canonical identification of vertical tangent spaces T^vP with the Lie algebra $\mathcal{G}$ of G). Note that it may be considered as a connection on M with holonomy group $\mathrm{Hol}(\nabla) \subset G$. Assume now that G is a reductive linear group. Then we may choose a G-invariant complement D to the subspace $\mathcal{G} \otimes T^* \cap T \otimes \Lambda^2 T^*$ into $T \otimes \Lambda^2 T^*$. A connection H in a G-structure π is called D-connection [Al-Mar] if the restriction of the structure function t to the submanifold $H(M) = \{H_p, p \in Q\} \subset J^1(Q)$ take values in D. Assume that G is a linear group of type one, that is the first prolongation

$$\mathcal{G}^{(1)} = \mathcal{G} \otimes T^* \cap T \otimes S^2 T^*$$

of its Lie algebra is zero. Then there exists a unique D-connection which is given by

$$Q \ni p \mapsto H_p = J_p^1(\pi) \cap t^{-1}(D).$$

Note that all linear Lie groups from Definition 1.1 are reductive. From classification of reductive linear Lie groups which have non-trivial first prolongation [Kob-Nag] it follows that all these groups with the exception of $GL(E) \otimes GL(H)$ has type 1 and the group $GL(E) \otimes GL(H)$ has type 2. We obtain

Corollary: *Any G-structure of Grassman type different from Grassman structure admit a unique D-connection.*

Recall that a G-structure is said to be 1-integrable if it admites a torsionless connection. Note that in this case any D-connection has no torsion.

2 Grassman type structures on manifolds of isotonic curves and sections

2.1 Isotonic bundles

Recall that any line bundle over projective line $\mathbb{C}P^1 = \mathbb{P}$ is isomorphic to the line bundle $\mathcal{O}(m) = \mathcal{O}(1)^m$ for some integer m where $\mathcal{O}(1)$ is the standard line bundle of hyperplanes with Chern class $c_1 = 1$. The authomorphism group $\mathrm{Aut}(\mathcal{O}(m))$ of the line bundle $\mathcal{O}(m)$, $m \neq 0$,is isomorphic to GL_2. It acts irreducibly on the space of sections

$$\Gamma\mathcal{O}(m) \simeq S^m\mathbb{C}^2 \simeq \mathbb{C}^{m+1}$$

for $m > 0$. By Grothendieck's Theorem any vector bundle N over $\mathbb{P}$ is isomorphic to a direct sum of line bundles that is

$$N \simeq r_1\mathcal{O}(a_1) \oplus \ldots \oplus r_k\mathcal{O}(a_k).$$

Definition: A holomorphic vector bundle N of rank r over projective line $\mathbb{P}$ is called isotonic with tone $m \in \mathbf{Z}$ if it is decomposed into a direct sum of r copies of the line bundle $\mathcal{O}(m)$:

$$N \simeq r\,\mathcal{O}(m). \tag{2.1}$$

Note that isotonic bundles are exactly semistable bundles over $\mathbb{P}$.

Lemma 2.1: *Let N be an isotonic vector bundle over $\mathbb{P}$ with tone m. Then the space ΓN of sections carries canonical Grassman structure*

$$\Gamma N = E \otimes H$$

$$E = \Gamma(N(-m)) \simeq \mathbb{C}^r, \quad H = \Gamma\mathcal{O}(m) \simeq \mathbb{C}^{m+1}$$

which is invariant under the action ρ of the automorphism group $Aut(N)$ on ΓN and

$$\rho(Aut(N)) = GL(E) \otimes SL_2^{[m]}, \tag{2.2}$$

where $SL_2^{[m]}$ is the irreducible linear group defined by the action of $SL_2 \subset Aut\,\mathcal{O}(m)$ on $\Gamma\mathcal{O}(m)$.

Proof: We can write $\Gamma N = \Gamma(N(-m) \otimes \mathcal{O}(m)) = \Gamma N(-m) \otimes \Gamma\mathcal{O}(m)$ where $N(-m) = N \otimes \mathcal{O}(-m)$. By (2.1) $N(-m)$ is a trivial bundle and, hence, $\Gamma N(-m) \simeq \mathbb{C}^r$.
From the isomorphism (2.1) , we derive that $\mathrm{Aut}(N) = GL_r \otimes SL_2^{[m]}$, where $SL_2^{[m]}$ is an extension of the action of SL_2 on $\mathbb{P}$ to an action on $r\mathcal{O}(m) \simeq N$ by automorphism. Since the group $GL_r \otimes SL_2^{[m]}$ acts irreducibly on ΓN, the existence of a unique invariant Grassman structure on ΓN follows from Lemma 1.1.

2.2 Isotonic curves

Definition: A rational curve P in a complex $(r+1)$-dimensional manifold Z is called isotonic with tone m (or m-isotonic for short) if its normal bundle $N_P = T_Z/T_P$ is isotonic that is

$$N_P \simeq r\,\mathcal{O}_P(m).$$

More precisely this means that for any parametrisation $f : \mathbb{P} \to P$, the bundle f^*N_P over $\mathbb{P}$ is isomorphic to $r\mathcal{O}(m)$.

Since any two parametrizations differ by an element from $\mathrm{Aut}(\mathbb{P}) = PL_2$, this definition is consistent. Moreover, in the space ΓN_P of sections there exists a canonical Grassman structure

$$\Gamma N_P = E_P \otimes H_P$$

given by Lemma 2.1.

Let $m \geq -1$. An isomorphism $\varphi : N_P \xrightarrow{\sim} r\mathcal{O}(m)$ of the normal bundle N_P onto the standard bundle $r\mathcal{O}(m)$ over $\mathbb{P}$ induces an isomorphism

$$\phi : \Gamma N_P \xrightarrow{\sim} \Gamma(r\mathcal{O}(m)) = \mathbb{C}^r \otimes \Gamma\mathcal{O}(m).$$

We say that ϕ is an *admissible coframe* of ΓN_P. The group

$$\mathrm{Aut}\,(r\mathcal{O}(m)) = GL_r \otimes SL_2^{[m]}$$

acts simply transitively on the set Q_P of admissible coframes by

$$\mathrm{Aut}\,(r\mathcal{O}(m)) \quad \ni \quad A: \qquad \phi \quad \mapsto \quad A\circ\phi.$$

Let

$$M \ni x \quad \mapsto \quad P(x) \subset Z$$

be a holomorphic family of rational curves in Z (in the sense of Kodaira) parametrized by an n-dimensional connected complex manifold M. Denote by $\mathcal{A} = \{(x,z) \in M \times Z,\ z \in M\}$ the incidence graph of the family. The natural projection

$$\mathcal{A} \to M$$

is a submersion by definition. It is locally trivial since all fibers are rational curves.

Assume now that all curves $P(x)$ are isotonic. Note that tone of $P(x)$ is equal to

$$m \;=\; \mu\left(N_{P(x)}\right) \;:=\; c_1\left(N_{P(x)}\right)/r, \qquad r \;=\; \mathrm{rank}\left(N_{P(x)}\right).$$

Hence, it doesn't depend on $x \in M$. Now we define Q as set of admissible coframes

$$\phi : \Gamma N_{P(x)} \xrightarrow{\sim} \mathbb{C}^r \otimes \Gamma\mathcal{O}(m).$$

in the space of sections of the normal bundle $N_{P(x)} = T_Z/T_{P(x)}$ for $x \in M$. The group $GL_r \otimes SL_2^{[m]}$ acts on Q freely with the orbit space M. From local triviality of the submersion $\mathcal{A} \to M$, it follows that $Q \to M = Q/GL_r \otimes SL_2^{[m]}$ is a (holomorphic) $GL_r \otimes SL_2^{[m]}$- principal bundle.

Moreover the (holomorphic) vector bundle

$$S: \quad M \ni x \quad \mapsto \quad \Gamma N_{P(x)}$$

is associated to the principal bundle Q and the action of $GL_r \otimes SL_2^{[m]}$ on $\mathbb{C}^r \otimes \Gamma\mathcal{O}(m)$. *We say that Q is the bundle of admissible coframes of the vector bundle S.*

Suppose now that the family $x \mapsto P(x)$ of isotonic curves is maximal at each point $x \in M$. Then by Kodaira's Theorem the vector bundle S is identified with the tangent bundle TM. Using the rigidity of the complex structure in $\mathbb{C}P^1$ and the normal bundle of isotonic curve and applying Kodaira's Theorem, we obtain

Proposition 2.2: *Let P is a isotonic curve with tone $m \geq -1$ in a $(r+1)$-manifold Z.*

Then there exist a maximal family $M \ni x \mapsto P(x)$ of m-isotonic curves which contains $P = P(x)$ and is parametrized by a connected $r(m+1)$-dimensional complex manifold M. Moreover, the germ of the family in P is unique up to an equivalence.

Moreover the manifold M carries natural $GL_r \otimes SL_2^{[m]}$-structure $Q \to M$.

In particular $TM = E \otimes H$, where E, H are vector bundles over M assosiated with Q and the standard irreducible representation of $GL_r \otimes SL_2^{[m]}$ in $\mathbb{C}^r$ and $\Gamma\mathcal{O}(m) = \mathbb{C}^{m+1}$, respectively.

Proof: (1) follows from Kodaira Theorem and the equalities

$$H^1(T_P) \quad = \quad H^1(\mathrm{End} N_P) \quad = \quad H^1(N_P) \quad = \quad 0.$$

Taking into account the last equality, Kodaira Theorem states that the tangent space $T_x M$ for any $x \in M$ is identified with $\Gamma N_{P(x)}$. Hence, the bundle $S = TM$ and the principal $GL_r \otimes SL_2^{[m]}$-bundle $Q \to M$ is a bundle of coframes of the tangent bundle. This implies (2) and (3).

2.3 Isotonic sections

Let $\pi : Z \to \mathbb{P}$ be a holomorphic fibration over the projective line and $\dim Z = r + 1$. A section $f : \mathbb{P} \to Z$ is called m-isotonic if the curve $f(\mathbb{P})$ is m-isotonic. As a corollary of Proposition 2.2 we obtain

Proposition 2.3: *Any m-isotonic section f of a fibration $\pi : Z \to \mathbb{P}$ is contained into a family of m-isotonic sections which is parametrized by an $r(m+1)$-dimensional manifold M.The manifold M carries natural right flat Grassman structure(that is a $GL_r \otimes 1_{m+1}$-structure).*

Proof: Since a sufficiently small deformation of the curve defined by a section remains a section, the first statement follows from Proposition 2.2. To prove the second, we identify the normal bundle N_f of a section from M with the bundle $f^* N_f$ over the projective line. Then we have

$$T_f M = \Gamma N_f = \Gamma(N_f(-m) \otimes \mathcal{O}_{\mathbb{P}}(m)) = \Gamma(N_f(-m)) \otimes \Gamma\mathcal{O}_{\mathbb{P}}(m) = E_f \otimes H,$$

where $H = \Gamma\mathcal{O}_{\mathbb{P}}(m)$ is a fixed vector space. Hence, the tangent bundle TM is identified with tensor product of the vector bundle $E : f \mapsto E_f$ and the trivial bundle H. This shows that M carries a right flat Grassman structure.