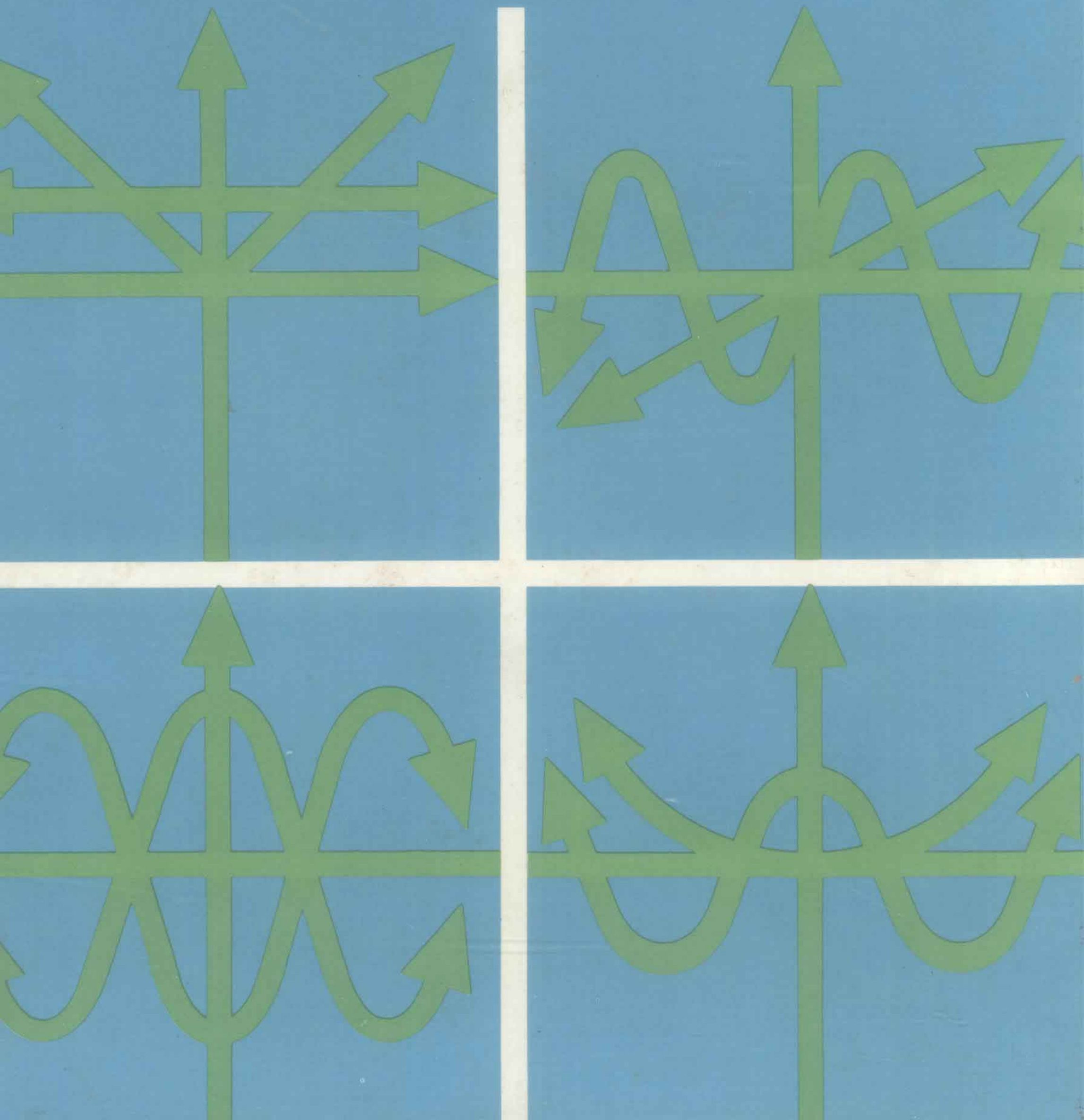
Franklin Wright/Kenneth E. Lindgren



ELEMENTARY FUNCTIONS

Pre-Calculus Mathematics

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Preface

This text has been written to prepare students for the study of calculus. It follows the CUPM recommendations for Mathematics 0 with the following exceptions. Functions of two variables are not included since they usually appear late in calculus and are discussed thoroughly there. Instead, sequences and series (Chapter 7) are presented to give the students a head start in understanding the limit concept which causes so much difficulty in the early stages of calculus. Also, permutations, combinations, and probability (Chapter 8) are included both to generate interest in applications and to provide a background for statistics.

The text has been classroom tested and has the following outstanding features.

1. Graphs are used throughout as a tool for understanding functions.
2. The slope function is used as an aid to graphing polynomial functions.
3. Trigonometry is developed with a wrapping function approach and proceeds to right triangle applications.
4. The text is written at a level easily read and understood by the students.
5. The exercises are many, varied, and meaningful.

Sufficient proofs are provided both in the explanations and the exercises to give the students a feeling for mathematical procedures but not so many as to produce "rigor" mortis. The techniques of reading and understanding theorems are too often delayed until calculus with the result that deciphering mathematical sentences occupies time the students need for understanding the ideas of calculus.

Functions are introduced as mappings in Chapter 1 with the domain and range concepts emphasized graphically. Functional addition, subtraction, multiplication, and division are discussed both algebraically and graphically. Composite functions with restricted domains and ranges are used to develop algebraic skills and to enforce the need for proper domains. Converses and a formal development of 1–1 functions lead into inverse functions.

Chapter 2 covers polynomial functions in general, then linear functions and quadratic functions in detail. Quadratic functions are related to the form $f(x) = a(x-h)^2 + p$ and the zeros are found graphically and with the quadratic formula. Synthetic division is used to evaluate polynomials after the Remainder Theorem and Factor Theorem are proved. The chapter is completed with a discussion of the Rational Zeros Theorem and upper and lower bounds for the zeros of polynomial functions.

In Chapter 3 an intuitive discussion of limits leads to continuity. The importance of continuity is emphasized with theorems on boundedness and absolute maximums and minimums on closed intervals. The slope function for polynomial functions is defined and used as a tool in graphing. No attempt is made to introduce calculus. Rational and algebraic functions are thoroughly discussed, including relating the graphs of functions by horizontal and vertical shifting. Complex numbers and the Fundamental Theorem of Algebra make the students aware of the zeros of functions in general.

The exponential functions, their graphs, and applications take up the first three sections of Chapter 4. The logarithm function is defined as the inverse of an exponential function, again with emphasis on 1–1 functions and on domains and ranges. Computations with common logarithms finish the chapter.

Chapters 5 and 6 provide the trigonometry needed for success in calculus. In Chapter 5 the circular functions are defined in terms of a wrapping function and the usual identities and reduction formulas are developed. The graphs of cos, sin, Cos^{-1}, and Sin^{-1} are discussed thoroughly. In Chapter 6 the trigonometric functions are defined in terms of circular functions and more identities are included along with right triangle applications. Other important topics are the Law of Cosines, the Law of Sines, and solving triangles.

Mathematical induction is the first topic in Chapter 7. Then, sequences are defined and convergent and divergent sequences are discussed. The sigma notation, Σ, and its properties are developed; and series are defined as sequences of partial sums.

Chapter 8 is not intended as a prerequisite for calculus and is optional for the course. The material on permutations, combinations, and probability is always interesting to students and is presented in a basic set theory approach that is easily read.

The answers to the odd-numbered exercises are contained in the back of the text and a solutions manual with sample tests and answers is available to the instructor. We wish to thank those students who suffered through the preliminary edition for their patience and suggestions and we thank our reviewers, Ross Beaumont, John Fujii, C. Louise Gillespie, and Del Hackert, for their many helpful ideas and corrections to the manuscript. We would like to thank Martha Allen for another outstanding editing job. Special accolades go to Pat Wright for a terrific job of typing and putting up with the eccentricities of two authors.

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Contents

1 Algebra of Functions

1.1 Sets, Real Numbers, and the Real Number Line

We will assume that the student is familiar with sets and their properties and with the real numbers and their properties. We will also assume that the student needs review and reinforcement of these ideas and we present this section for that purpose.

The term "set" will be considered an undefined term or, in other words, a term so basic that any attempt to define it will only involve terms and ideas more complicated than set itself.

Braces, { }, are used to denote sets, and the objects that belong to a set are called *elements* or *members* of the set. The elements can be listed, as $\{a, b, c, d\}$, or indicated by a rule, as {the first four letters of the alphabet}. The symbol "$\in$" is used to indicate that an element "belongs to" or "is an element of" a set, and "$\notin$" indicates that an element "does not belong to" or "is not an element of" a set. Thus, if $A = \{f, g, h\}$, then $f \in \{f, g, h\}$ (or $f \in A$) and $k \notin A$.

Consider two sets, say A and B: if every element of A is an element of B, then A is a *subset* of B (symbolized $A \subset B$ and read "A is a subset of B" or "A is contained in B"). For example, if $A = \{2, 4, 6, \ldots, 50\}$ and $B = \{1, 2, 3, 4, \ldots, 50\}$, then $A \subset B$. But, if $C = \{0, 1, 2, 3, \ldots, 10\}$, then $C \not\subset B$ (read "C is not a subset of B") since $0 \in C$ but $0 \notin B$. Also, $B \not\subset C$ and $B \not\subset A$ since B has elements not in C and not in A.

A set important in any discussion of sets is the set without any elements, symbolized by $\varnothing$ or $\{\ \ \}$ and called the *empty set* or *null set*. (SPECIAL NOTE: The symbols $\{\varnothing\}$ and $\{0\}$ do *not* denote the empty set. The set $\{\varnothing\}$ has the empty set as an element, and $\{0\}$ has the number 0 as an element.) *If A represents any set, then* $\varnothing \subset A$. This fact can be proven in the following way. Either $\varnothing \subset A$ or $\varnothing \not\subset A$. If $\varnothing \not\subset A$, then $\varnothing$ must have some element not in A. But $\varnothing$ has no elements and therefore does not have any elements that are not in A. The only conclusion is that $\varnothing \subset A$.

The following notation will be used for the various sets of numbers indicated.

Natural Numbers $N = \{1, 2, 3, 4, 5, \ldots\}$
Whole Numbers $W = \{0, 1, 2, 3, 4, 5, \ldots\}$
Integers $Z = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$

The set of integers includes all the whole numbers and their opposites (or negatives).

Rational Numbers $Q = \{$all numbers of the form a/b where $a, b \in Z$ and $b \neq 0\}$

Rational numbers are simply fractions whose numerators and denominators are integers, with the restriction that no denominator can be 0. One important fact, whose proof appears in the Appendix, is that any rational number can be written as an *infinite repeating decimal*. For example, $\frac{1}{3} = 0.3333\ldots$ and $\frac{5}{11} = 0.454545\ldots.$

Irrational Numbers $H = \{$all numbers that can be written as infinite nonrepeating decimals$\}$

The irrational numbers deserve special comment. Their description or definition says that they *can be* written as infinite nonrepeating decimals, not that they *must be* written so. In fact, irrational numbers such as $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, $-\sqrt{3}$, $7^{2/3}$, $2 + \sqrt{2}$, $5\sqrt{11}$, $\cos 45°$, and π are seldom written in their decimal form. For example, $\pi = 3.14159\ldots$† and

†π has been carried out to 100,265 decimal places and no pattern of repeating digits has been found. This was done by Daniel Shanks and John W. Wrench, Jr. on an IBM 7090 machine in New York City on July 29, 1961. The calculation took 8 hours, 43 minutes and the check took 4 hours, 22 minutes. It was estimated that the same calculation would take 30,000 years for a mathematician with a 10-place electric desk computer.

$$\pi = 3.141592653589793\ldots \text{(omitting next 99,975 places)} \ldots 5493624646\ldots$$

The fact that π can be proven irrational assures us that there will be no repeating pattern of digits.

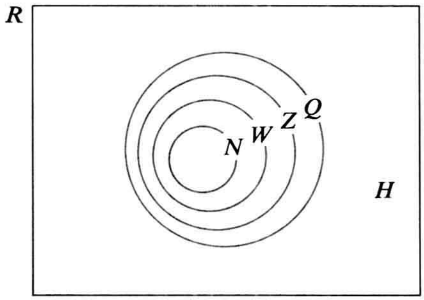
$\sqrt{2} = 1.4142\ldots$; however, the decimal form is usually only approximated or rounded off as $\pi \approx 3.14$ and $\sqrt{2} \approx 1.414$. Rational numbers are treated in the same way in their decimal form: $\frac{1}{7} = 0.142857142857\ldots$ but we would probably use 0.143 as $\frac{1}{7}$ rounded off to the nearest thousandth. The student must understand that irrational numbers are just as useful and important as rational numbers.

Real Numbers $R = \{\text{all numbers that are rational or irrational}\}$

Thus, in set notation, we see:

$$\varnothing \subset N \subset W \subset Z \subset Q \subset R \text{ and } H \subset R.$$

The following Venn diagram also illustrates these relationships.



Two sets can be put into one-to-one correspondence (abbreviated 1–1 correspondence) if each element in each set can be paired with one element in the other set with no elements left over. As a simple example, the sets $\{a, b, c\}$ and $\{1, 2, 3\}$ can be put into 1–1 correspondence as illustrated:

$$\begin{array}{ccc} \{a, & b, & c\} \\ \updownarrow & \updownarrow & \updownarrow \\ \{1, & 2, & 3\} \end{array}$$

However, $A = \{1, 2, 3, \ldots, 10\}$ and $B = \{2, 3, 4, \ldots, 10\}$ cannot be put into 1–1 correspondence because, no matter how the elements are paired, one element in A will be left over. In a more sophisticated example, the set $N = \{1, 2, 3, 4, 5, \ldots\}$ and the set $E = \{2, 4, 6, 8, 10, 12, \ldots\}$ can be put into 1–1 correspondence by matching each element $n \in N$ with $2n \in E$.

$$\begin{array}{ccccccc} \{2, & 4, & 6, & 8, & 10, \ldots, & 2n, & \ldots\} \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ \{1, & 2, & 3, & 4, & 5, \ldots, & n, & \ldots\} \end{array}$$

A geometric 1–1 correspondence can be set up between the points on two line segments even though the segments are of different lengths (Figure 1–1). In the figure, think of a pendulum swinging through point P.

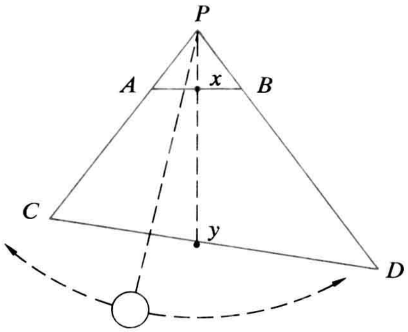


Figure 1-1

As the pendulum swings, each point x of segment $\overline{AB}$ can be paired with one point y of $\overline{CD}$ and vice versa. Thus, we have illustrated a 1-1 correspondence between the points on the segments $\overline{AB}$ and $\overline{CD}$.

We say that the real numbers can be put in a 1-1 correspondence with the points on a straight line. This means that for every real number there is a corresponding point (called the graph of the number) on a straight line and for every point on the straight line there is a real number (called the coordinate of the point). Given a straight line, we choose some point, any point, on the line to correspond to 0. We choose another point, usually to the right of 0 if the line is horizontal, to correspond to 1 (Figure 1-2).

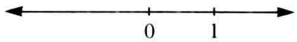


Figure 1-2

Once these two points have been chosen, then the 1-1 correspondence between R and the points on the line is determined (Figure 1-3).

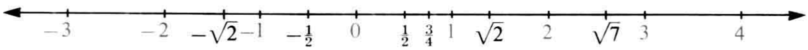


Figure 1-3

To illustrate the fact that irrational numbers, such as $\sqrt{2}$, $\sqrt{3}$, and $\sqrt{5}$, do have a place on the number line, we make use of the Pythagorean Theorem and Figure 1-3, as shown in Figure 1-4.

PYTHAGOREAN THEOREM In a right triangle, the sum of the squares of the two sides is equal to the square of the hypotenuse.

Of special interest is the concept of an *interval* of real numbers. For this we need set builder notation and inequalities. The notation $\{x \mid \ldots\}$

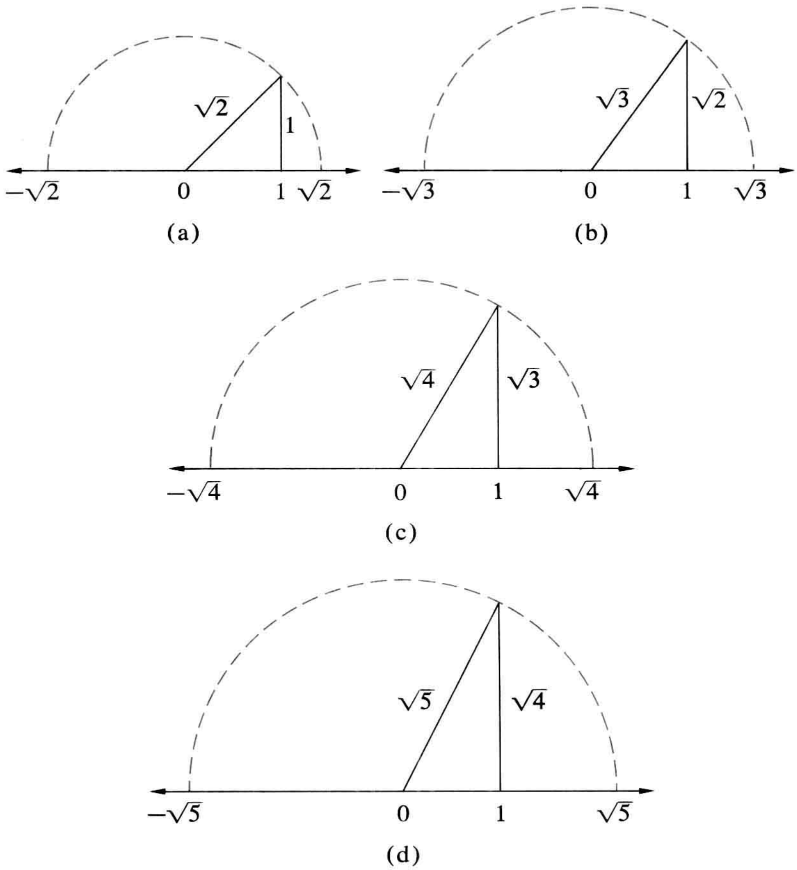


Figure 1-4

is read "the set of all x such that . . . ." For example, $\{x \mid x^2 - 2x + 1 = 0\}$ is read "the set of all x such that $x^2 - 2x + 1 = 0$." The symbol "$<$" is read "is less than" when the inequality is read from left to right and "is greater than" when it is read from right to left. For example, $-4 < 3$ is read "-4 is less than 3" or "3 is greater than -4." Similarly, "$>$" is read "is greater than," when read from left to right, and "is less than," when read from right to left.

Thus, combining inequalities with set notation, $\{x \mid 5 < x \leq 7\}$ is read "the set of all x such that x is greater than 5 and x is less than or equal to 7."

An *interval* is (a) the set of *all* real numbers that lie between two real numbers called *end points of the interval,* or (b) the set of all real numbers greater than some real number or less than some real number, or (c) the set of all real numbers. The end points may or may not be included in an interval, as illustrated below.

Suppose $a, b, x \in R$ and $a < b$:

Name	Description	Graph
Open interval	$\{x \mid a < x < b\}$ or $x \in (a, b)$	
Closed interval	$\{x \mid a \le x \le b\}$ or $x \in [a, b]$	
Half-open interval	$\{x \mid a < x \le b\}$ or $x \in (a, b]$	
Half-open interval	$\{x \mid a \le x < b\}$ or $x \in [a, b)$	
Open interval	$\{x \mid a < x\}$ or $x \in (a, \infty)$	
Half-open interval	$\{x \mid x \le b\}$ or $x \in (-\infty, b]$	

An open interval includes no end point; a closed interval includes two end points; a half-open interval includes only one end point. The notations

$$(a, b), \quad [a, b), \quad (a, b], \quad \text{and} \quad [a, b]$$

should not be confused with the ordered pair notation the student has seen before and which will be discussed in Section 1.2. The meaning of the notation should be clear from the context of the material. Also, the symbol ∞, read "infinity," does not represent a particular number. Its use simply means that the numbers under consideration have no upper bound.

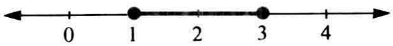
Two common set operations (methods of combining sets to form new sets) are *union* and *intersection*. We will give their formal definitions, then illustrate them with intervals of real numbers.

DEFINITION 1.1-1 The *union* of two sets, say A and B, is the set of all elements that belong to A *or* to B. (The union is symbolized $A \cup B$, read "A union B.")

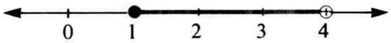
The word "or" in the definition is used in the inclusive sense. That is, if an element belongs to either one set or the other or even to both, then it belongs to the union.

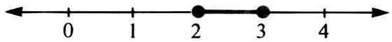
DEFINITION 1.1-2 The *intersection* of two sets, say A and B, is the set of all elements that belong to both A *and* B. (The intersection is symbolized $A \cap B$, read "A intersect B.")

Example 1 Suppose $A = \{x \mid 1 \le x \le 3\}$ and $B = \{x \mid 2 \le x < 4\}$. Find $A \cup B$ and $A \cap B$ and graph both sets.

SOLUTION: A 

B 

$A \cup B = \{x \mid 1 \le x < 4\}$ 

$A \cap B = \{x \mid 2 \le x \le 3\}$ 

Example 2 Suppose $C = \{y \mid \sqrt{2} < y\}$ and $D = \{y \mid y \le 5\}$. Find $C \cup D$ and $C \cap D$ and graph both sets.

SOLUTION: C 

D 

$C \cup D = \{y \mid y \in R\}$

$C \cap D = \{y \mid \sqrt{2} < y \le 5\}$

DEFINITION 1.1-3 If two sets, say A and B, are such that $A \cap B = \varnothing$, then A and B are *disjoint*.

Example 3 Suppose $E = \{x \mid x \in (-\sqrt{2}, 0)\}$ and $F = \{x \mid x \in [0, 1)\}$. Show that E and F are disjoint.

SOLUTION: E 

F 

$E \cap F = \varnothing$; therefore E and F are disjoint.

Exercises 1.1

List the elements or members of the following described sets.

1. {the integers between -3 and 4}
2. {the natural numbers between -8 and 3}
3. {the positive integers ending with the digit 4}
4. {the real numbers satisfying $3x - 5 = 2$}
5. {the irrational numbers satisfying $x^2 = 3$}
6. {the rational numbers satisfying $4x^2 - 1 = 0$}

Represent the following sets using set builder notation.

7. $\{-3, -2, -1, 0, 1\}$
8. $\{103, 104, 105, 106\}$
9. $\{1, 3, 5, 7, 9, 11, \ldots\}$
10. $\{\ldots, -4, -2, 0, 2, 4, \ldots\}$
11. $\{1, \sqrt{2}, \sqrt{3}, 2, \sqrt{5}, \ldots\}$
12. $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\}$

Verify that each of the following sets is empty.

13. $\{x \mid x^2 = 3, x \in Q\}$
14. $\{x \mid 3x - 1 = 0, x \in Z\}$
15. $\{x \mid 2x + 6 = 0, x \in W\}$
16. $\{x \mid x^2 + 1 = 0, x \in R\}$
17. $\{x \mid x < -2, x \in N\}$
18. $\{x \mid 2^x = -4, x \in R\}$
19. $\{x \mid 3^x = -9, x \in R\}$

20. From plane geometry, let:
    (a) $P = \{x \mid x \text{ is a parallelogram}\}$
    (b) $Q = \{x \mid x \text{ is a quadrilateral}\}$
    (c) $R = \{x \mid x \text{ is a rectangle}\}$
    (d) $S = \{x \mid x \text{ is a square}\}$
    (e) $T = \{x \mid x \text{ is a trapezoid}\}$
    (f) $U = \{x \mid x \text{ is a rhombus}\}$

    Determine which sets are subsets of the others.

Describe the following sets.

21. {the set of all points in a plane a given distance from a fixed point}
22. {the set of all points in a plane a given distance from a given line}
23. {the set of all points in a plane equidistant from two distinct fixed points}
24. {the set of all points in space a given distance from a fixed point}
25. {the set of all points in space a given distance from a given line}
26. {the set of all points in space equidistant from two distinct fixed points}

Graph the following sets on a number line.

27. $\{\ldots, -3, -2, -1, 0\}$
28. $\{-\sqrt{37}, -\sqrt[3]{27}, -\frac{5}{3}, 0, \sqrt{1}, \sqrt{4}, \sqrt{10}, \sqrt{48}\}$
29. $\{-\sqrt{42}, -\sqrt{9}, -1, \frac{9}{2}, \sqrt{28}, \sqrt{34}, \sqrt[3]{343}\}$
30. $\{x \mid x \in R\}$
31. $\{x \mid x \in [-2, 3], x \in R\}$
32. $\{x \mid x \in [4, 83), x \in R\}$
33. $\{x \mid x \in (-\sqrt{5}, \sqrt{3}), x \in R\}$
34. $\{x \mid x \in (-5, 3], x \in W\}$
35. $\{x \mid x \in Q\}$
36. $\{x \mid x \in H\}$

Graph the following sets of real numbers on a number line.

37. $\{x \mid -1 < x < 2\} \cap \{x \mid 0 < x < 3\}$
38. $\{x \mid -1 < x < 2\} \cup \{x \mid 0 < x < 3\}$
39. $\{y \mid -3 \leq y < 0\} \cup \{y \mid 0 < y \leq 5\}$
40. $\{y \mid -3 \leq y < 0) \cap \{y \mid 0 < y \leq 5\}$