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WORKING ANALYSIS

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Working Analysis

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Preface

Why Another Advanced Calculus Book?

Over the years, I have become more and more dissatisfied with our advanced calculus course. In most books used for this type of course, theorems are proved to prove more theorems. An "application" of a theorem is either a trivial calculation or a piece of the proof for another theorem. There are no examples or exercises that use the methods of analysis to solve a real problem. The traditional advanced calculus course has little or no contact with the world outside of mathematics.

A second shortcoming of the traditional advanced calculus course is that, in spite of the abundance of high-quality mathematical software, this kind of course does not use the computer. Although the most important function of an advanced calculus course is to teach students rigorous methods of proof, the computer can be used to illustrate many important concepts. Rates and orders of convergence of sequences, error in various approximations, and numerical differentiation and integration are but a few. Furthermore, many practical problems cannot be considered solved until some numbers are produced. The work of the applied analyst does not end with an existence theorem.

My goals in writing this book are to teach the techniques and results of analysis and to show how they can be applied to solve problems. I have gone outside of the usual range of applications in physics to include examples from biology, sociology, chemistry, and economics.

Prerequisites

I assume that the students using this book have had three semesters of calculus (including multivariable) and one semester of linear algebra. It would be helpful if the students have some experience with a software package. The course will be enhanced if students can graph functions of one and two variables and make simple computations. However, this is not a course in programming. I usually provide the students with MATLAB codes for functional iteration and a Newton solver in one and several variables. A (rather crude) code for the method of steepest descent is also provided. These codes can be found on my web page, `www.math.umd.edu/~jec`.

The Text

The text contains more than enough material for a two semester course. Part I is an introduction to analysis in one dimension. Part II, for the second semester, deals with functions of several variables. Some books begin such a course immediately with several variables, but I feel this approach is too difficult for most students. As a result, there is some repetition of material in Parts I and II. Nevertheless, seeing some ideas (like that of a contraction mapping) in one, and then in several variables, helps in their assimilation by students. Examples and problems that treat applications are sprinkled throughout the text, but the more detailed discussions are found in Part II, which deals with functions of several variables. Exercises of a computational nature are found in almost every chapter, but the more challenging ones are found in the later chapters.

Summary of Part I

Chapter 1 is a brief treatment of the foundations of analysis. One of the reasons analysis is difficult for students is that it involves manipulation of inequalities. I have set aside a section, 1.3, that deals with this issue. Chapter 2 is devoted to sequences, and particular attention is paid to the fact that some sequences converge faster than others. Numerical examples are given and estimates are made to illustrate the difference between linear and quadratic convergence. The examples foreshadow the discussion in Chapter 6 of functional iteration and Newton's method.

Chapter 3 deals with limits of functions and continuity. After establishing that a continuous function on a closed bounded interval must attain its extreme values, I introduce the method of golden-section search to locate the maximizer and minimizer. It is important for the student to realize that there are many optimization problems where the function may not be differentiable or where the derivative is very difficult to calculate. The method of bisection is used to prove the intermediate value theorem.

Linear approximation is the theme of Chapter 4. The O and o notations are introduced to describe the error in linear approximation. In computational exercises, students calculate the errors to see how rapidly they decrease. The mean value theorem is discussed and used to prove a one-dimensional version of the inverse function theorem. The latter is used to justify a change of variable in a differential equations example. l'Hôpital's rule and the second derivative test are followed by an example from economics to close the chapter.

In Chapter 5, higher derivatives and Taylor polynomials are discussed. In several examples, the Lagrange form of the remainder is used to derive useful approximations and to estimate their error. Another use of the Taylor approximation is to derive expressions for numerical differentiation. Centered difference quotients are then used to discretize a nonlinear differential equation. The resulting nonlinear system of equations will be solved later in Part II using Newton's method. Finally, polynomial interpolation with an expression for the error is discussed. These results are used later in Chapter 7 to derive quadrature rules. More immediately, they are used in an introduction to convex functions.

Chapter 6, *Solving Equations in One Dimension*, is devoted to the contraction mapping theorem and to Newton's method. Functional iteration is easy to discuss in one dimension, and it provides a simple computational tool that does not use the derivative. It also provides the central idea in the proof of the inverse function theorem in several dimensions, proved in Part II. The proof of Newton's method shows how quadratic convergence arises, and prepares the student for the treatment in higher dimensions. These topics are also natural situations for computational exercises.

The treatment of the Riemann integral in Chapter 7 is fairly standard. To keep matters simple and practical, I have not stated the most general forms of the fundamental theorem of calculus. The simple quadrature rules are derived using polynomial interpolation. In particular, I think I have given a more conceptual explanation of why Simpson's rule works better than expected; it does, in fact, arise from interpolating by a cubic polynomial. I have also given examples using the change of variable formula for integrals that are of use in numerical calculation. In the exercises for Section 7.4, the student can do certain numerical experiments and then use analysis to explain the observed behavior.

Many ideas from earlier chapters come together in Chapter 8. The all-important concept of uniform convergence is used to justify term-by-term integration and differentiation. To show a use of the latter, I obtain a power series expansion for solutions of the Airy equation. The chapter ends with the example $f(x) = \exp(-1/x^2)$, which is C^∞ but not analytic at $x = 0$. In the exercises, the student is asked to graph the derivatives of f to see if he or she can explain how this function can have a convergent power series expansion for $x \neq 0$ but not at $x = 0$. I think this is an excellent example of the use of the computer in this type of course.

The elementary functions $\sin x, \cos x, e^x$, and $\log x$ are used freely in examples. These functions are defined, and their basic properties are developed in the Appendix to Part I.

Summary of Part II

Because I am eager to get to applications, the discussion of norms, topology, and continuity in Chapter 9 is rather brief. Rather than introduce the idea of a metric space, I prefer to deal with three different, and very useful, norms on $\mathbb{R}^n$.

In Chapter 10, the idea of linear approximation is again emphasized, this time for functions of several variables and for vector-valued functions. It is one of the important themes of the course. I do not always state and prove theorems in the most general form. I find that sometimes stronger hypotheses make for a simpler and more transparent proof.

Chapter 11 picks up the material of Chapter 6 and extends it to systems of equations. After proving the contraction mapping theorem in several variables, I use the idea behind Newton's method to prove a simplified form of the Kantorovich existence theorem. Students should learn not only existence theorems, but also constructive algorithms that can be used for numerical calculation. The interplay between the inverse function theorem, or the implicit function theorem, and the calculation of a family of solutions that depend on a parameter is very important. Several examples are explored to show that the "big" theorems can be employed to further the analysis. The examples and the exercises sometimes require the use of mathematical software.

Chapter 12 deals with problems in unconstrained optimization. In addition to the usual second derivative test, we include a short section on convex functions in several variables. This is followed by some algorithms for finding a minimum that do not use the second derivatives. The method of steepest descent has a simple proof of convergence and is easy to visualize. Unfortunately, it is not very efficient, and it is wrong to leave the student with the impression that this method is used in practice. For this reason, I have added a section on conjugate gradient methods. Instructors who feel that this topic takes the course too far in the direction of numerical analysis can state the main idea and then move on. The problems that arise in fitting data with curves that depend in a nonlinear fashion on the parameters are very interesting, and they are important in applications. The Gauss-Newton algorithm is appealing in its simplicity, and it is widely used.

Constrained optimization is the subject of Chapter 13. The Lagrange multiplier method is a good example in which an abstract theorem (the implicit function theorem) is used to derive a system of equations that one can try to solve by the methods of Chapter 11. In applications, there are often one or more parameters present in the problem, and it is important to see how the solutions of the Lagrange equations depend on these parameters. The example of the three bar linkage illustrates how the number of constraints depends on the choice of variables used to describe the problem and how the solutions depend on the parameters present. We also derive the analogue of the second derivative test for constrained maxima and minima. The Karush-Kuhn-Tucker conditions for constraints with inequalities are needed for many problems in economics, in particular the Averch-Johnson effect. The analysis of this problem uses the ideas of Chapter 11 as well.

In Chapter 14, the elements of integration in several variables are treated rather briefly, relying on the treatment in one variable, which the student has already seen. Numerical methods for integrating functions of two and three variables are a very appropriate use of Riemann sums and are easily derived from the standard methods in one variable. The main emphasis in Chapter 14 is on the change of variable theorem. It is an important tool in the numerical evaluation of integrals. It is also useful in finding the probability density function of combinations of random variables.

Chapter 15 explores ways in which the integration theory of Chapter 14 can be applied to partial differential equations. After giving a criterion that justifies differentiation under the integral, we investigate the convolution. The solutions of the diffusion equation can be represented in terms of a convolution, and we explore their properties. We also use convolution to prove the Weierstrass polynomial approximation theorem. Finally, we use the change of variable theorem in a key way to derive the Euler equations of fluid flow.

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Of course, final responsibility for correctness of the text is mine. Readers who find errors or unduly confusing passages are encouraged to notify me at my e-mail address. I will post errata and clarifications on my webpage.

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Part I