A WORKBOOK IN MATHEMATICAL METHODS FOR ECONOMISTS

R. QUENTIN GRAFTON

TIMOTHY C. SARGENT

A WORKBOOK IN MATHEMATICAL METHODS FOR ECONOMISTS

R. Quentin Grafton
University of Ottawa

Timothy C. Sargent
University of Ottawa

THE McGRAW-HILL COMPANIES, INC.

New York St. Louis San Francisco Auckland
Bogotá Caracas Lisbon London Madrid
Mexico City Milan Montreal New Delhi San Juan
Singapore Sydney Tokyo Toronto

McGraw-Hill
A Division of The ***McGraw-Hill*** *Companies*

A Workbook In Mathematical Methods For Economists

1 2 3 4 5 6 7 8 9 0 QPD QPD 9 0 9 8 7

ISBN 0-07-007122-5

The editor was Lucille Sutton;
the production supervisor was Diane Dilluvio.
The cover was designed by Christopher Brady.
Quebecor-Dubuque was printer and binder.

http://www.mhcollege.com

To Carol-Anne and Anik

Preface

This workbook is written for all those people who have tried problems in mathematics and felt the frustration of not obtaining the correct answer and not knowing where they went wrong. The workbook is structured on the belief that for most people mathematical techniques can only be learned through the use of examples and by independently working through problems and verifying the solutions. Readers who *first* attempt the problems and then verify their answers will be able to master many of the mathematical methods needed to study intermediate and advanced undergraduate economics.

Our goal is to provide the reader with the resources to become proficient in the mathematical analysis used in economics and business. The self-study approach of the workbook is well-suited to students taking courses in mathematical economics and economic theory and who wish to practise the techniques and methods learned in class. The workbook is also suitable for readers who may have previously learned the topics presented in the workbook but who need a review and practice. To help the reader, each chapter includes a review and introduction of the techniques used in the questions. The review is *not* a substitute to consulting the recommended texts because space limitations prevent the inclusion of almost all the theorems and proofs that underlie the techniques.

We assume the reader has a basic knowledge of algebra and the rules of differentiation. As much as is possible, each chapter is structured to stand alone and, where necessary, references to other chapters are provided. In general, the questions are more difficult at the end than at the beginning of each chapter. We recommend, therefore, that the reader answer the lower numbered problems first so as to build the confidence and skills required for the more advanced questions.

An important feature of the workbook is the emphasis on objective-based learning. To this end, we list the skills that will be acquired by correctly answering the questions given in each chapter. This provides a ready-made structure for self-learning and enables those readers who only wish to master sub-topics within a chapter to consult the relevant questions.

We are grateful for financial support from the University of Ottawa in preparing the manuscript and for the facilities provided by the Norwegian School of Economics and Business Administration (NHH) and the Centre for Fisheries Economics during the summer of 1993. We also thank Stein Ivar Steinshamn of NHH and Chantale Lacasse and David Gray of the University of Ottawa for valuable comments on various chapters. We are especially indebted to Greg Flanagan of Mount Royal College for his encouragement and his many suggestions which have greatly improved the workbook. We would also like to thank students at the University of Ottawa for finding errors in the manuscript and for their many useful questions and comments. The efforts of Milan Jayasinghe and Sean Maguire who provided editorial assistance, Greg Flanagan who helped prepare the camera-ready copy, and Erin Flanagan who prepared the index are much appreciated.

R.Q.G and T.C.S
Ottawa, Canada
August, 1996

Contents

Chapter 1

Introduction

Mathematics and Economics

Economics involves the observation of real phenomena, their characterization into a simplified or abstract view of the world and then an analysis and interpretation of potential outcomes. Such a process is called model building by economists and is used to examine positive ("what is") and normative ("what should be") questions about the world around us.

Model building involves abstract representations and can be greatly helped by the language of mathematics. The analysis of economic models may also be improved by the use of mathematical techniques. Thus, a good understanding of mathematics is very useful to economists. Mathematics, however, does not tell economists what are the important economics problems that should be studied nor does it suggest what variables should be the focus of examination or what assumptions are appropriate. Mathematics, therefore, should always be the servant of economics and not *vice versa.* This perspective may seem surprising to the reader who works through the problems in the workbook as the focus is on mathematical technique. Our objective, however, is simply to provide a resource to students so that they can master the mathematical techniques often used in economics. In mastering these techniques we hope that the reader can apply her knowledge and intuition of economics to better address the economic problems that we face.

The exercises in the workbook cover a wide range of applications in economics. Three concepts that underlie the techniques are: Equilibrium; Optimization; Rationality.

Equilibrium. The concept of equilibrium in economics is one of a state of rest where variables will not, in general, change unless there is some perturbation to the system.

Optimization. The notion of optimization is that individuals and firms are economic agents and will always try to do the best they can given their constraints. Thus, if a firm wishes to maximize its profits given a set of market conditions and technology the firm will act in such a way that it does maximize its profits.

Rationality. A related concept is rationality in that economic agents act in a logical and consistent fashion that conforms with their assumed objectives.

All three concepts are a simplification of reality and how people behave but provide the rationale for the use of many of the techniques used by economists. For example, the assumption that agents are rational and optimize is necessary when using calculus to solve for the demand functions of individuals.

Outline of the Workbook

This book is not a text in economic theory, economic modeling or rigorous mathematics. It is, no more and no less, a workbook in the mathematical methods used in economics and business. The reader should use the workbook as a supplement to any text used in courses in mathematical economics and for topics in microeconomic and macroeconomic theory. The workbook should be viewed as a resource that can be used to master the mathematical techniques that are an integral part of economic analysis. To this end, the workbook uses objective-based learning to provide structure to the readers self-study of the topics. By consulting the objectives found at the beginning of each chapter, the reader can identify what is to be learned from each question and, if necessary, focus on sub-topics.

Unlike some texts in mathematical methods for economics and business, we actively encourage the reader to consult the many good references on the various topics. At the back of each review section a list of references is provided that should be consulted to understand the theory and proofs behind the techniques and to progess to more advanced topics. In addition, other books are recommended for further practice in problem solving and for applications of the techniques in economics and business. Readers who understand the concepts in the recommended references and correctly answer the questions in the workbook should be confident they have mastered the material.

The workbook follows the topics that are normally covered in the many texts on mathematical methods for economists. Chapter 2 of the workbook uses matrix algebra to examine a number of important economic questions. A large class of economic problems involves systems of linear equations. Such systems of equations can be analyzed and a solution found or characterized using matrices. For example, input-output analysis has been used to examine structural problems in economies and to provide guidance to policy makers where appropriate. Such analysis is greatly helped by a knowledge of matrix algebra. Matrix algebra is also useful in determining whether the solutions to problems are a maximum or a minimum and, thus, are widely used in economics. There are several good texts that can be consulted on matrix algebra. Glaister [19] (chapters 1-10) is particularly useful and provides programs in BASIC to solve problems in matrix algebra. Haeussler and Paul [21] (chapter 6) and Chiang [13] (chapters 4 and 5) are also good references and provide a number of examples and solved problems. We also highly recommend Sydsæter and Hammond [32] (chapters 12-14). Dowling [17] (chapters 10-12) and Toumanoff and Nourzad [34] (chapter 8) also provide many worked examples of how matrices may be applied in economics while Rowcroft [26] (chapter 15) provides an excellent discussion on input-output analysis.

Chapter 3 covers many topics and asks the reader to characterize various functions and sets and use some important theorems and results. The material is fundamental to using mathematics in economics. For example, the ability to identify whether a constraint set is convex or whether the objective function is concave can greatly simplify the solution to a large number of problems. The notion of homogeneity is also widely applied in production and consumer theory and the Taylor series and the implicit function theorem are useful techniques for solving certain economic problems. There are many texts that cover the various topics in the chapter. For the reader who needs a review of algebra and equations before attempting the problems, we recommend Haeussler and Paul [21] (chapters 1 and 2). Dowling [17] (chapters 1 and 2) is also a useful guide to the terminology of mathematics and provides a very good review of differentiation and the use of derivatives in economics (chapters 3 and 4). We also recommend Toumanoff and Nourzad [34] (chapters 3-5) for a good review of differentiation and differentials and their applications to economics. Glaister [19] (chapter 11) provides a good introduction to set theory and its applications in economics while Chiang [13] (chapters 6 and 7), Bressler [10] (chapter 6), Ostrosky and Koch [25] (chapter 3) and Sydsæter and Hammond [32] (chapters 6 and 7) are good references on limits, continuity and differentiation.

The concept of optimization is directly applied in chapters 4-8. Chapter 4 examines the problem of optimization without constraints while chapter 5 addresses this question when an agent's behaviour is constrained. Constrained optimization refers to a wide variety of problems including an individual maximizing her utility subject to a budget constraint or a firm maximizing profits subject to a given production technology. Particularly useful references on unconstrained optimization include Birchenhall and Grout [6] (chapter 5), Chiang [13] (chapter 12), Archibald and Lipsey [1] (chapters 6 and 7), Glaister (chapter 15), Baldani et al. [2] (chapters 7 and 8) and Sydsæter and Hammond [32] (chapter 9 and 17). For constrained optimization we recommend Chiang [13] (chapter 12), Birchenhall and Grout [6] (chapter 8), Archibald and Lipsey [1] (chapters 10 and 11), Baldani et al. (chapters 11 and 12) and Sydsæter and Hammond [32] (chapter 18).

When the solution of constrained optimization problems are substituted into the original objective function of the agent one obtains the optimum value or indirect objective function. An integral part of microeconomics is duality theory which uses the relationship between the optimal value function and the objective function to solve a wide variety of problems. Chapter 6 uses a number of important results from duality theory to solve consumer and producer problems. A valuable reference on duality results and how they may be obtained using the envelope theorem is presented in Silberberg [29] (chapter 7). A useful introduction to duality with worked examples is Birchenhall and Grout [6] (chapters 10-12) while Dixit [15] (chapter 5) and Sydsæter and Hammond [32] (chapter 18.7-18.8) offer a good overview of maximum value functions and the envelope theorem. Baldani et al. [2] also provides a number of applications using duality theory and the envelope theorem (chapter 14).

An optimization problem characterized by an objective function and constraints that are linear in the unknown variables is called linear programming and can be solved using a powerful algorithm, the simplex method. Chapter 7 uses the simplex method to solve for a number problems that arise in business and addresses some of the difficulties when the problem is misspecified. Definitive references on linear programming include Bradley et al. [9], Dorfman et al. [16] and Wu and Coppins [38]. Dorfman et al. [16], in particular, provides many applications of linear programming to economics.

Linear programming is a subset of nonlinear programming where the unknown variables may be nonlinear in the objective function and/or constraints. In reality, most of the optimization problems actually solved by people or firms are nonlinear problems. Chapter 8 uses techniques, such as the Kuhn-Tucker conditions, to solve for demand functions when an individual faces inequality constraints. There are many references on nonlinear programming. We recommend Lambert[24] (chapter 5), Chiang (chapter 21), and Beavis and Dobbs [4] (chapter 2).

Many questions in economics are dynamic or involve optimization over time. A basis for understanding dynamic problems is a knowledge of integral calculus presented in chapter 10. Integration involves finding a primitive function from a derived function. This is useful, for example, in finding the total variable cost function from a marginal cost function or to calculate the area beneath a demand function so as to determine an individual's consumer surplus. Integration is also widely used in econometrics which uses statistics to address economic problems. Useful references on integration include Haeussler and Paul [21] (chapters 16 and 17), Chiang [13] (chapter 13), Sydsæter and Hammond [32] (chapter 10) and Holden and Pearson [23] (chapter 6). Dowling [17] (chapters 16 and 17) also provides numerous worked examples in integral calculus.

Dynamic analysis is examined in chapters 11 and 12 on difference and differential equations. Both techniques are useful in characterizing the time paths of economic variables. These techniques are particularly valuable in macroeconomics which is often concerned with the concept of equilibrium and how variables change over time. Chiang [13] (chapters 14-18), and especially Sydsæter and Hammond [32] (chapters 20 and 21), provide a good introduction to the topics while Dowling [17] (chapters 18-20) provides a number of worked examples. In more complex dynamic problems, the time paths of variables may not always be characterized by real numbers. For this reason, questions are provided in chapter 9 so as to provide the reader with the skills necessary to characterize more complicated problems using complex numbers. A definitive reference on complex numbers is Sydsæter [31] (chapter 2).

To get the most out of the workbook you must *first* attempt the questions before looking at the solutions. It is only by working through problems that you will be able to master the concepts and techniques. It is has been our experience that students who look at the solutions before attempting the questions rarely master the material. Only after you have obtained a solution or are unable to make any further progress on a problem should you consult the solutions. To build your skills progressively, the more difficult questions are located towards the end of each chapter.

There are many good computer programs available that can solve most of the problems in the workbook. We strongly believe, however, that to learn many mathematical techniques it is important to know how to solve problems without the aid of a computer. Thus, we recommend that you first solve the problems in the workbook by hand and then compare your answer with our solutions and the solutions you may obtain using a computer.

Message to the Instructor

The table of contents provides a ready guide to instructors who wish to have their students learn, review and apply particular tools of mathematical analysis. A glossary of over 200 definitions is also provided at the back of the workbook for students wishing to know the meaning of the many terms used in mathematical economics.

The workbook has been used as a supplementary text in a first and second course in mathematical economics. Chapters 2-4 are used in the first course and chapters 5-12 in the second course. Students are assigned problems as the topics are covered in class and are expected to work through the problems. Students of all abilities greatly appreciate the opportunity to work through questions themselves and to correct and learn from their own mistakes. Where necessary, additional problems have also been assigned to provide further practice to the students.

In addition to courses in mathematical economics, chapters 2 and 10 have been assigned to students requiring a review in matrix algebra and integration and who are enrolled in courses in econometrics and statistics. The objectives at the start of the chapter help students identify gaps in their knowledge which can be remedied by doing the problems and consulting the relevant texts.

Other chapters of the book may also be used in second and third courses in microeconomics and macroeconomics. Chapter 5—optimization with equality constraints—was written for and has been used by students in first and second courses in microeconomics. This chapter provides a review of the method of Lagrange and gives students the opportunity to solve economic problems with equality constraints. Chapter 6 on duality theory and chapter 8 on nonlinear programming are particularly useful for students in a second and third course in microeconomics.

Finally, the workbook would also be a valuable text for students taking a review course in mathematical economics in graduate programs in economics. Parts of various chapters should also prove useful in other courses in economics and business.

Chapter 2

Matrix Algebra

Objectives

The questions in this chapter will help the reader to master the fundamentals of matrix algebra. Readers who can correctly answer all the questions should be able to:

1. Determine whether the columns of a matrix are linearly independent (Question 3).
2. Determine the rank of a matrix (Questions 2, 3 and 4).
3. Solve for the inverse of a matrix (Questions 1, 4 and 5).
4. Solve simple problems in input-output analysis (Question 5).
5. Solve systems of linear simultaneous equations (Questions 4, 5 and 6) using elementary row operations, the method of the inverse, and Cramer's rule.
6. Find eigenvalues and eigenvectors (Question 7).
7. Determine the definiteness of a quadratic form (Question 8).

Review

Matrices occur frequently in economics. Many economic models can be expressed as systems of linear equations, and matrix algebra provides efficient methods for solving such problems. Indeed, most of the chapters in the handbook make use of matrix algebra in one form or another.

A matrix consists of numbers and/or variables called *elements* arrayed in a particular way. Suppose, for example, that we have the following system of equations:

$$\begin{aligned} 3x_1 + 4x_2 &= -7y_1 + 2y_2 \\ 4x_1 &= 9y_2. \end{aligned}$$

This system can be written more compactly as

$$\mathbf{Ax} = \mathbf{By}$$

where we define the matrices in upper case letters as $\mathbf{A}$ and $\mathbf{B}$ by

$$\mathbf{A} = \begin{pmatrix} 3 & 4 \\ 4 & 0 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} -7 & 2 \\ 0 & 9 \end{pmatrix};$$

and the vectors in lowercase letters $\mathbf{x}$ and $\mathbf{y}$ by

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \text{ and } \mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

The *elements* of a matrix can be described by the row and column where they are located. Thus, $b_{12} = 2$ represents the element in the first row and second column of the matrix $\mathbf{B}$ while $a_{11} = 3$ is the element in the first row and first column of matrix $\mathbf{A}$. The *dimension* or *order* refers to the number of rows and columns of a matrix. For example, a matrix with 2 rows and 3 columns has the dimension (2×3). The matrices $\mathbf{A}$ and $\mathbf{B}$ have the dimension (2×2).

A *transpose* of a matrix $\mathbf{A}$ is the matrix rewritten such that first, second, $\cdots$ rows become the first, second, $\cdots$ columns. The transpose of a matrix $\mathbf{A}$ is defined as $\mathbf{A}^T$. Thus for the matrix $\mathbf{B}$ defined above, its transpose is:

$$\mathbf{B}^T = \begin{pmatrix} -7 & 0 \\ 2 & 9 \end{pmatrix}$$

A *square* matrix is a matrix where the number of rows equals the number of columns. The *main diagonal* of a square matrix is the elements formed by drawing a line from the upper left corner to the bottom right corner. Thus, the main diagonal of the matrix $\mathbf{A}$ above are the elements 3 and 0. An *upper triangular* matrix is a square matrix where all the elements to the left of the main diagonal equal zero. A *lower triangular* matrix is a square matrix where all the elements to the right of the main diagonal are zero. A *symmetric* matrix is a square matrix where $a_{ij} = a_{ji}$ for $i \neq j$.

A vector is a matrix with only one row or column. The *length* of a vector $\mathbf{x}$ is denoted by $|\mathbf{x}|$ and is computed using the *Euclidean formula*

$$|\mathbf{x}| = \sqrt{x_1^2 + x_2^2 + \cdots x_n^2}$$

where n is the dimension of the vector. A vector of unit length has a length of 1.

A set of vectors is said to be *linearly dependent* if one of the vectors can be expressed as a linear combination of the others. For example, consider the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$, where:

$$\mathbf{v}_1 = \begin{pmatrix} 4 \\ 2 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 1 \\ 0.5 \end{pmatrix}$$

These two vectors are linearly dependent because $\mathbf{v}_1 = 4\mathbf{v}_2$.

More formally, a set of m vectors $\mathbf{v}_1,\ldots,\mathbf{v}_m$ is linearly dependent if $\sum_{i=1}^m c_i\mathbf{v}_i = 0$ and $c_i \neq 0$ for at least one c_i. A set of vectors is *linearly independent* if none of the vectors can be expressed as a linear combination of the others, so that $\sum_{i=1}^m c_i\mathbf{v}_i = 0$ is *only* true when each scalar $c_i = 0$.

A useful way to test for linear dependence is to write the vectors as a system of linear equations in c_i. Consider the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ above. They will be linearly dependent if there exists $(c_1, c_2) \neq (0, 0)$ such that:

$$c_1 \begin{pmatrix} 4 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 0.5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This condition can be restated as the following two equations:

$$\begin{aligned} 4c_1 + c_2 = 0 &\Rightarrow 4c_1 = -c_2 \\ 2c_1 + 0.5c_2 = 0 &\Rightarrow 2c_1 = -0.5c_2 \end{aligned}$$

Substituting the second equation into the first we obtain

$$c_2 = c_2$$

This is true for any value of c_2 and not just $c_2 = 0$. Thus, the vectors $\mathbf{v_1}$ and $\mathbf{v_2}$ are linearly dependent.

Two vectors are *orthogonal* if the vectors are perpendicular (at right angles) to each other. Formally, this definition requires that $\mathbf{v}_1^T \cdot \mathbf{v}_2 = 0$, where the dot operator '·' is defined as the *inner product*: the sum of the product of the corresponding elements in each vector. Using the definitions of $\mathbf{v}_1$ and $\mathbf{v}_2$ from above, we find that

$$\mathbf{v}_1^T \cdot \mathbf{v}_2 = \begin{pmatrix} 4 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0.5 \end{pmatrix} = (4 \times 1) + (2 \times 0.5) = 5$$

and the two vectors are therefore not orthogonal. A set of vectors which are mutually orthogonal and where each vector is of unit length is an *orthonormal* set. A vector can be *normalized* to be of unit length by dividing each element of the vector by its length.

Matrices can be added and subtracted provided that they have the same dimension. Note that it is not necessary that the matrices be square. The procedure for addition (subtraction) is to add (subtract) the corresponding element in the second matrix to (from) the first matrix. For example, consider the matrices:

$$\mathbf{C} = \begin{pmatrix} 12 & 5 & -3 \\ 0 & 2 & 1 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} 6 & 6 & -2 \\ -1 & 0 & 1 \end{pmatrix}.$$

Then

$$\mathbf{C} - \mathbf{D} = \begin{pmatrix} 12-6 & 5-6 & (-3)-(-2) \\ 0-(-1) & 2-0 & 1-1 \end{pmatrix} = \begin{pmatrix} 6 & -1 & -1 \\ 1 & 2 & 0 \end{pmatrix};$$

Matrices can be multiplied by a number, in matrix terminology, a scalar. In this case, every element in the matrix is multiplied by this number. Matrices can also be multiplied by another matrix provided the two matrices are *conformable*, i.e., that the number of columns in the premultiplying matrix equals the number of rows in the postmultiplying matrix. The product of the two matrices has a dimension equal to the number of rows in the premultiplying matrix and the number of columns in the postmultiplying matrix. Matrix multiplication is illustrated as follows where $\mathbf{A}$ premultiplies $\mathbf{D}$:

$$\mathbf{AD} = \begin{pmatrix} (3 \times 6) + (4 \times -1) & (3 \times 6) + (4 \times 0) & (3 \times -2) + (4 \times 1) \\ (4 \times 6) + (0 \times -1) & (4 \times 6) + (0 \times 0) & (4 \times -2) + (4 \times 1) \end{pmatrix} = \begin{pmatrix} 14 & 18 & -2 \\ 24 & 24 & -8 \end{pmatrix};$$