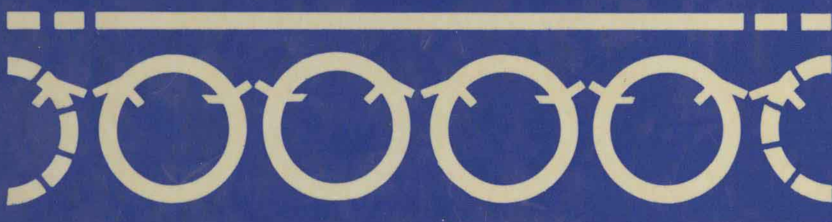
CONVECTION AND CHAOS IN FLUIDS

J. K. Bhattacharjee



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'Taking Three as the subject to reason about -
  A convenient number to state -
We add Seven, and Ten, and then multiply out
  By One Thousand diminished by Eight.

'The result we proceed to divide, as you see,
  By Nine Hundred and Ninety and Two:
Then subtract Seventeen, and the answer must be
  Exactly and perfectly true......'

- Lewis Carroll

To

Ujjaini

FOREWORD

Complicated behaviour of simple systems has made it possible to construct models for various hydrodynamic instabilities. These model systems usually consist of a set of nonlinear coupled differential equations or nonlinear maps. The relevance of these systems to actual hydrodynamics, as demonstrated by recent experiments, makes it possible to discuss hydrodynamic instabilities at a fairly elementary level. These notes are an attempt at making the complex behaviour of fluids accessible to senior undergraduate students.

Fully developed turbulence which permits no simplification has been virtually omitted. By dealing exclusively with viscous fluids, the vast literature of conservative dynamical systems has not been touched upon. The available analytic approximation techniques (perturbation theory in one form or another) for hydrodynamic flows and the relevance of certain prototype dissipative dynamical systems to the study of hydrodynamic instabilities form the core of the book.

I have benefitted from teaching a one-semester course on "Order and chaos in nature" at IIT, Kanpur and from a series of lectures at the University of Manchester.

Jayanta K. Bhattacharjee

CONTENTS

Chapter I

ONSET OF CONVECTION

Convection in fluids was studied from a quantitative angle in the early nineteen hundreds by Lord Rayleigh. The arrangement consisted of a liquid confined between two parallel plates and heated from below. For convenience in setting up the theory it is desirable to consider the plates to be infinite in extent. In practice the typical lateral dimension (L) of the plates is always finite. To satisfy the theoretical idealisation one needs to have $L >> d$, the separation between the plates. It is well known that the fluid heated from below does not begin convecting unless a critical temperature difference ΔT_c is established between the plates. The hydrostatically unstable situation of density increasing upwards is stabilised by viscosity and thermal conductivity until the density gradient is large enough to cause the upper layers to tumble down and the hotter lower layers to rise. Instead of ΔT (the temperature difference between the plates), one usually talks about a dimensionless number R, called the Rayleigh number, defined as

$$R = \frac{\alpha(\Delta T)gd^3}{\lambda\nu} , \qquad (1.1)$$

where α is the thermal expansion coefficient, λ is the thermal diffusivity, ν is the kinematic viscosity and g is the acceleration due to gravity. Obtaining the critical Rayleigh number for the onset of convection will be the purpose of this chapter.

We begin by writing down the hydrodynamic equations that govern the velocity ($\vec{v}$) and temperature (T) fields of the fluid. The velocity field satisfies the Navier-Stokes equation

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v}.\vec{\nabla})\vec{v} = -\frac{\vec{\nabla}P}{\rho} + \vec{g} + \nu\nabla^2 \vec{v}, \qquad (1.2)$$

where P is the pressure and ρ is the density. The temperature field satisfies a diffusion equation

$$\frac{\partial T}{\partial t} + (\vec{v}.\vec{\nabla})T = \lambda\nabla^2 T . \qquad (1.3)$$

We notice that these equations admit a steady conduction solution if $\vec{v} = 0$ and

$$T = T_s = T_1 + \frac{T_2 - T_1}{d} Z = T_1 - \frac{\Delta T}{d} Z . \qquad (1.4)$$

Here we have taken the plate separation to be in the z-direction and the infinite lateral extent of the plates makes the problem one-dimensional. T_1 is the temperature of the lower plate and T_2 is that of the upper one. The pressure and density in the steady state are P_s and ρ_s and have to satisfy

$$\frac{\vec{\nabla}P_s}{\rho_s} = \vec{g} . \qquad (1.5)$$

The question we need to ask now is whether this solution for the conduction state is stable. To test the stability of this state against convection we imagine that small perturbations $\delta\vec{v}$, δT, $\delta\rho$ and δP are made about the conduction state solution and study the equations governing the time development of the perturbations, taking care to linearize the equations in the perturbations. If the resulting time dependence of the perturbations is such that they decay exponentially in time, then the unperturbed state is stable. If on the other hand the perturbations do not die out, then the original state is unstable against the perturbation and an instability sets in. This is the principle of linear stability analysis and this is what we shall now try to implement.

At this point it is useful to introduce the Boussinesq approximation. In this approximation one considers the fluid incompressible except when dealing with the buoyancy term which drives the convection. Thus $\delta\rho$ appears only in the coefficient of $\vec{g}$ and can be set equal to zero everywhere else. In particular this means that the velocity field satisfies the incompressibility condition in the form

$$\vec{\nabla}.\ \vec{v} = 0 \quad . \tag{1.6}$$

The perturbation $\delta\vec{v}$ satisfies this constraint since the steady state velocity is zero. Thus

$$\vec{\nabla}.\ \delta\vec{v} = 0\ . \tag{1.7}$$

The linearized version of Eq.(1.2) now reads

$$(\frac{\partial}{\partial t} - \nu\ \nabla^2)\delta\vec{v} = - \frac{\vec{\nabla}(\delta P)}{\rho_s} + \frac{(\vec{\nabla}P_s)\delta\rho}{\rho_s{}^2} = -\frac{\vec{\nabla}(\delta P)}{\rho_s} + \vec{g}\ \frac{\delta\rho}{\rho_s}\ . \tag{1.8}$$

Taking the divergence of either side leads to

$$\nabla^2\ \delta P = (\vec{g}.\vec{\nabla})\delta\rho = -\ g\ \frac{\partial(\delta\rho)}{\partial Z}\ , \tag{1.9}$$

when use is made of Eq.(1.7). The partial derivative with respect to z leads to

$$\nabla^2\ \frac{\partial\,\delta P}{\partial Z} = -\ g\ \frac{\partial^2\delta\rho}{\partial Z^2}\ . \tag{1.10}$$

Denoting the components of $\delta\vec{v}$ by v_x, v_y and v_z, we have from Eq.(1.8), for the z-component of the velocity

$$(\frac{\partial}{\partial t} - \nu\nabla^2)\ v_z = -\ \frac{1}{\rho_s}\ \frac{\partial}{\partial Z}\ \delta P\ -\ g\ \frac{\delta\rho}{\rho_s}. \tag{1.11}$$

Taking the laplacian,

$$\nabla^2\ (\frac{\partial}{\partial t} - \nu\nabla^2)v_z = -\ \frac{1}{\rho_s}\ \nabla^2\ \frac{\partial(\delta P)}{\partial Z} - \frac{g}{\rho_s}\ \nabla^2\delta\rho$$

$$= -\ \frac{g}{\rho_s}\ (\nabla^2 - \frac{\partial^2}{\partial Z^2})\delta\rho = -\ \frac{g}{\rho_s}\ (\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2})\ \delta\rho, \tag{1.12}$$

having used Eq.(1.10) in the second step. Using the definition of the thermal expansion coefficient

$$\alpha = -\frac{1}{\rho}\frac{\partial\rho}{\partial T}, \tag{1.13}$$

we can rewrite Eq.(1.12) as

$$\nabla^2\left(\frac{\partial}{\partial t} - \nu\nabla^2\right)v_z = \alpha g\left(\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2}\right)\delta T\ . \tag{1.14}$$

Turning to Eq.(1.3), the linearized version is easily seen to be

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \lambda\nabla^2\right)\delta T &= -(\delta\vec{v}\,.\vec{\nabla})\,T_s \\ &= v_z\,\frac{\Delta T}{d}\ .\end{aligned} \tag{1.15}$$

We scale all lengths by d, time by d^2/ν, the field δT by ΔT and the velocity field by λ/d. The dimensionless quantities are defined as

$$(X,Y,Z)/d = (x,y,z)\ , \tag{1.16a}$$

$$\tau = t\nu/d^2, \tag{1.16b}$$

$$\theta = \delta T/\Delta T\,, \tag{1.17a}$$

$$w = v_z d/\lambda\ , \tag{1.17b}$$

$$\sigma = \nu/\lambda\ , \tag{1.17c}$$

and in terms of these definitions Eqs.(1.14) and (1.15) become

$$\nabla^2\left(\frac{\partial}{\partial\tau} - \nabla^2\right)w = R\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\theta \tag{1.18}$$

and

$$\left(\sigma\frac{\partial}{\partial\tau} - \nabla^2\right)\theta = w\ . \tag{1.19}$$

We now seek solutions to the above equations which correspond to the formation of convection rolls. The rolls would be characterised by a two-dimensional wave number in the x-y plane with components k_x and k_y and hence the space-time dependent solutions would be of the form

$$\theta(x,y,z,t) = \Theta(z)\, e^{i(k_x X + k_y Y) + p\tau} , \qquad (1.20)$$

$$w(x,y,z,t) = W(z)\, e^{i(k_x X + k_y Y) + p\tau} , \qquad (1.21)$$

where $\Theta(z)$ and $W(z)$ satisfy (from Eqs.(1.18) and (1.19))

$$(D^2 - a^2)(D^2 - a^2 - p)W = Ra^2\Theta , \qquad (1.22)$$

$$(D^2 - a^2 - \sigma p)\Theta = -W, \qquad (1.23)$$

with $D = \frac{d}{dz}$ and 'a', the dimensionless wave number, given by

$$a^2 = (k_x^2 + k_y^2)d^2 . \qquad (1.24)$$

Eqs.(1.22) and (1.23) are eigenvalue equations for the relaxation rate p. If the resulting p is such that $Re(p) < 0$, fluctuations decay to zero as $t \to \infty$ and the initial state is stable. If however $Re(p) > 0$, the initial conduction state is unstable against the "roll" solution. The perturbations grow in time and are eventually arrested due to the nonlinear terms.

Before proceeding further, we need to specify the boundary conditions on $W(z)$ and $\Theta(z)$. Since the temperatures of the top and bottom plates are maintained*, the temperature perturbations vanish at z=0 and 1. Since the plates are stationary, we must have W=0 at z=0 and 1. If the plates are rigid, then the x and y components of the dimensionless velocity field (u and v respectively) too vanish at the boundaries. This implies that $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial y}$ are zero on the surfaces z=0 and 1. The continuity condition now leads to $\frac{dW}{dz} = 0$ on z=0 and 1. Thus the rigid boundary conditions are

* We assume high thermal conductivity for the plates.

$$W = DW = \Theta = 0 \quad \text{on } z = 0 \text{ and } 1. \tag{1.25}$$

For analytic calculations it is often convenient to use free boundaries. This implies that there are no stresses on the horizontal surfaces, leading to

$$\frac{\partial u}{\partial z} = - \frac{\partial w}{\partial x} = 0 \tag{1.26}$$

and

$$\frac{\partial v}{\partial z} = - \frac{\partial w}{\partial y} = 0 \tag{1.27}$$

at z=0 and 1, since on the plates, w = 0 independent of x and y. The continuity equation now yields

$$D^2 W = 0 \ . \tag{1.28}$$

Thus the free boundary conditions are

$$W = D^2 W = \Theta = 0 \quad \text{on } z = 0 \text{ and } 1. \tag{1.29}$$

We first treat the free boundary conditions. It is clear immediately that solutions satisfying the boundary conditions can be written as

$$W = A_n \sin(n\pi z) \tag{1.30}$$

$$\Theta = B_n \sin(n\pi z) \ , \tag{1.31}$$

where n is an integer. Self-consistent determination of A_n and B_n lead to the eigenvalue equation

$$\text{Det} \begin{vmatrix} (n^2\pi^2 + a^2)^2 + p(n^2\pi^2 + a^2) & - Ra^2 \\ 1 & -(n^2\pi^2 + a^2) - \sigma p \end{vmatrix} = 0 \tag{1.32}$$

To obtain the critical value of R, we find the value R_o corresponding to p=0 as

$$R_o = \frac{(n^2\pi^2 + a^2)^3}{a^2} . \qquad (1.33)$$

It is clear from the roots of the quadratic in p(Eq(1.32)) that for $R < R_o$, p is always negative and hence the perturbations decay in time, while for $R > R_o$, p is positive, leading to instability. The fact that the roots are real is related to the theorem that the instability is stationary. The critical value R_c of R is now determined by finding the minimum of R_o. Hence we need n=1 and have to minimize the resulting R_o as a function of 'a'. This leads to

$$R_c = \frac{27\pi^4}{4} \qquad (1.34)$$

and

$$a_c^2 = \frac{\pi^2}{2} . \qquad (1.35)$$

Thus we find the value of the Rayleigh number at which convection begins and the wavelength of the convection cells. Notice that the theory does not tell us what the shape of the cells will be — that is determined by the x and y components of the wavenumber vector. The critical Rayleigh number is determined by the magnitude of the wavenumber alone and hence is not affected by the shape.

In an actual laboratory experiment, the rigid boundary conditions are the natural ones. So we really need to find R_c by enforcing the boundary conditions of Eq.(1.25). To do this we eliminate Θ from Eqs.(1.22) and (1.23) and obtain

$$(D^2-a^2)(D^2-a^2-p)(D^2-a^2-\sigma p)W = -Ra^2W . \qquad (1.36)$$

For p=0, the condition for critical R, we have

$$(D^2-a^2)^3 W = -Ra^2W \quad , \tag{1.37}$$

with the boundary conditions

$$W = DW = 0 \quad \text{on } z = 0 \text{ and } 1 \tag{1.38}$$

and

$$(D^2-a^2)^2W = 0 \text{ on } z = 0 \text{ and } 1, \tag{1.39}$$

the last boundary condition following from Eq.(1.22) with $\Theta = 0$ on the boundaries. The solutions of Eq.(1.37) are going to be symmetric or antisymmetric about the centre plane $z = 1/2$. The symmetric solutions correspond to lower values of R. The even solution for W can be written as

$$W = A_1 \cosh \gamma_1 (z - \tfrac{1}{2}) + A_2 \cosh\gamma_2 (z - \tfrac{1}{2}) + $$

$$+ A_3 \cosh \gamma_3 (z - \tfrac{1}{2}) \tag{1.40}$$

where γ_1, γ_2 and γ_3 are the three roots of

$$(\gamma^2 - a^2)^3 + Ra^2 = 0 \tag{1.41}$$

and A_1, A_2 and A_3 are three undetermined constants to be fixed from the boundary conditions. Using the boundary conditions of Eqs.(1.38) and (1.39), we find three homogeneous equations in A_1, A_2 and A_3. The solvability condition determines R_o as a function of 'a' and the minimization with respect to 'a' leads to

$$R_c \simeq 1708 \tag{1.42}$$

and

$$a_c \simeq 3.12, \tag{1.43}$$

significantly different from the free boundary values of Eqs. (1.34) and (1.35). The experimental data are in good agreement with the above values of R_c and a_c.

We now show how closed form analytic approximants to the critical values of R_c and a_c quoted in Eqs. (1.42) and (1.43) can be obtained. We return to Eqs. (1.22) and (1.23) and make use of the fact that the instability is going to be stationary and set p=0 in determining R_o. We look for the symmetric solutions and make a change of origin so that z ranges from -1/2 to 1/2. The solution for Θ can then be expanded in a cosine series as

$$\Theta(z) = \sum_{n=0}^{\infty} A_n \cos (2n+1)\pi z . \tag{1.44}$$

With this expansion for $\Theta(z)$ inserted in Eq.(1.22), we can solve for W(z) under the boundary conditions that W = DW = 0 at z = 1/2. Returning to Eq.(1.23) and demanding consistency leads to determination of R_o in terms of 'a'. Minimization with respect to 'a' yields R_c.

We illustrate with a single-mode truncation for $\Theta(z)$, i.e. keep only the n=0 term in the expansion of Eq.(1.44). Inserting $\Theta(z) = A_o \cos\pi z$ in Eq.(1.22) and solving for W(z), we get for the even solutions

$$W(z) = A \cosh az + Bz \sinh az - \frac{R_o A_o a^2}{(\pi^2 + a^2)^2} \cos\pi z . \tag{1.45}$$

The boundary conditions yield

$$A \cosh \frac{a}{2} + \frac{B}{2} \sinh \frac{a}{2} = 0 \tag{1.46a}$$

and

$$a A \sinh \frac{a}{2} + B\left(\frac{a}{2} \cosh \frac{a}{2} + \sinh \frac{a}{2}\right) = - \frac{\pi R_o A_o a^2}{(\pi^2 + a^2)^2} \tag{1.46b}$$

determining A and B in terms of A_o, R_o and 'a'. Inserting