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H Brezis, M G Crandall &
F Kappel (Editors)

Semigroups, theory and applications

VOLUME I

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F Kappel (Editors)
Université Pierre et Marie Curie / University of Wisconsin-Madison / Karl-Franzens-Universität Graz

Semigroups, theory and applications

VOLUME I

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Semigroups, theory and applications

VOLUME I

Preface

The Autumn Course of 1984 held at the International Centre for Theoretical Physics (ICTP) in Miramare (Trieste, Italy) during the period November 12 to December 14, 1984, was devoted to the theme "Semigroups, Theory and Applications". In accordance with the basic aim of the ICTP to promote scientific maturity of developing countries the structure of the course was the following: The program of the first three weeks consisted of basic courses in order to provide an introduction to various aspects of the area for participants with limited background. In the fourth week more advanced courses were devoted to topics of current research and the fifth week had the character of a conference on evolution problems. The course received considerable interest as is documented by the number of approximately 90 participants from developing countries.

This volume contains manuscripts for most of the talks presented during the fifth week of the course. Unfortunately due to other obligations some of the speakers could not provide a contribution to these proceedings. Another volume in this series will contain a representative part of the lectures presented during the first four weeks of the course.

Of course, it is our obligation to thank the funding agencies of the ICTP (Italian Government, UNESCO and IAEA) which made this course possible. We also immensely acknowledge support provided by the staff of the ICTP. Moreover, Prof. Abdus Salam, director of the ICTP, underlined the importance of this type of enterprises through his constant visible interest during the course. Professors L. Bertocchi, A.M. Hamende and H. Talafi contributed in various stages of the organization. We especially appreciate support by Prof. G. Vidossich during the preparation of the course.

The success of the course could not have been possible without the joint efforts by the lecturers, speakers and participants. Finally we want to express our thanks to Mrs. Bridget Buckley (Pitman) and to Dr. W. Dietl (IAEA), who by joint efforts made publication of these proceedings in the

Research Notes possible. The excellent typing of the manuscripts for this volume was done by Mrs. G. Krois (Graz), who also provided efficient secretarial support concerning publication of these proceedings.

December 1985

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Non-monotone semilinear elliptic equations

Introduction

We study the problem

$$(P)\quad \begin{cases} -\Delta u = f(u) + g & \text{on } \Omega \quad u > 0 \quad \text{on } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

Ω is an open subset of $\mathbb{R}^N$, g a non negative function on Ω, f is defined on $\Omega \times \mathbb{R}$ ($f(u)(x) = f(x,u(x))$) and satisfies:

$$(H1)\quad \begin{cases} x \to f(x,r) & \text{measurable, for all } r \in \mathbb{R}^+ \\ r \to f(x,r) & \text{is convex, non decreasing and } f(x,0) \equiv 0 \text{ on } \Omega. \end{cases}$$

Let $G(x,y)$ be the Green function of the Dirichlet problem. If v is the solution of

$$\begin{cases} -\Delta v = h & \text{on } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases}$$

we have $v(x) = \int_\Omega G(x,y)h(y)dy$ which we denote by $v = Gh$.

Let u be a measurable non negative function on Ω, finite allmost everywhere on Ω, such that:

$$u(x) = \int_\Omega G(x,y)f(y,u(y))dy + Gg(x) \quad \text{for a.e. } x \in \Omega. \tag{1}$$

We call it an <u>integral solution</u> of (P). It is in some sense the weakest definition of a solution of (P).

Our purpose is first to give a necessary and sufficient condition on g which insures the existence of an integral solution of (P). This result allows us to define a norm, denoted by $|g|$, and can be summarized in the following way, (P) has an integral solution if and only if $|g| \leq 1$. Thus, in order to have a solution, g must be regular enough (if not we have $|g| = +\infty$) and small enough.

This result provides us with a criterion on g for the existence of a solution of (P) in a weakest sense. Does this solution have some regularity when g is supposed regular? For instance, does an integral solution of (P) is bounded when g is bounded and $|g| \leq 1$? In this case, we deduce that it is a classical solution which satisfies the boundary condition in a classical sense. This property does not hold in the general case. Indeed, we can find a counter example in [5]. When Ω is a ball of $\mathbb{R}^N$ and $N \geq 10$, they prove that the problem

$$\begin{cases} -\Delta u = \lambda e^u & \text{on } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has a bounded non negative solution if and only if $\lambda < \lambda^*$, where $\lambda^* > 0$. When $\lambda \uparrow \lambda^*$, we can easily deduce from our existence result and from the definition of an integral solution that the least classical solution of this problem converges to the least integral solution of the limit problem. We obtain a least integral solution of a problem (P) (with $f(x,r) = \lambda^*(e^r - 1)$ and $g = \lambda^*$) which is unbounded although g was bounded.

However, we prove the following result: under some suitable hypothesis on f which does not restrict its growth at the infinity, if we suppose $|g| < 1$ then, roughly speaking, $\left|\frac{u}{Gg}\right|$ is bounded on Ω. Here, u is the least integral solution of (P). Thus, if $|g| < 1$, the least integral solution has all the regularity that we can expect.

What happens when $|g| = 1$? The counter example before proves that this estimate fails in general. First, recall the situation when we are able to obtain a good estimation on all the solutions of (P) (when $f(r) = r^\gamma$ with $\gamma < \frac{N+2}{N-2}$ and $N < 10$ for instance, see [4]). The problem

$$(P_\lambda) \begin{cases} -\Delta u = f(u) + \lambda g & \text{on } \Omega \\ u > 0 & \text{on } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where $f \in C^2(\mathbb{R})$ and satisfies (H1) and g smooth enough, has a classical solution if and only if $\lambda \leq \lambda^*$ where $\lambda^* > 0$ (ours results imply that $|\lambda^* g| = 1$, thus $\lambda^* = 1$).

If the solution remains bounded in $\lambda = \lambda^*$, it is proved that

$$(P') \begin{cases} -\Delta v = f'(u)v & \text{on } \Omega \quad v > 0 \quad \text{on } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases}$$

has also a solution ([4]). It is one of the main argument to prove the existence of a turning point on the branch of solution of (P_λ). Our last result is to prove that this property remains true even if the non linearity of f forbides a good estimate in $\lambda = \lambda^*$. Indeed, we prove that if Ω is bounded, $Gg \in L^\infty(\Omega)$ and $|g| = 1$, (P') has still a solution under the same assumption on f that before.

These results will be proved in [1]. The similar results hold for parabolic equations and are established in [2].

The second part of this paper is devoted to the existence result. In the third part, we present the estimation on u/Gg and in the last one, we precise our result in the case $|g| = 1$.

1. Existence result for (P)

Notations: $L_0^\infty(\Omega)$ = {measurable bounded non negative function with compact support in Ω}.

$f^*(x,r)$ denotes the conjugate function of $f(x,r)$

that is: $f^*(x,r) = \sup_{\alpha \geq 0} r\alpha - f(x,\alpha)$ a.e. $x \in \Omega$.

We suppose:

(h) there exist $c > 1$ and $h \in L_0^\infty(\Omega)$, $h \not\equiv 0$ such that $f^*(c\frac{h}{Gh})Gh \in L^1(\Omega)$.

Then we have (let us recall that (H1) is given in the introduction)

Theorem 1 (see [3]). Assume (H1), (h) and $u_0 \geq 0$, $u_0 \in L^1_{loc}(\Omega)$ there exists a non negative measurable function u such that

$$u = G(f(u)) + u_0, \; uh \in L^1(\Omega)$$