LECTURES ON VON NEUMANN ALGEBRAS

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Preface

This book is based on lectures delivered in July-August 1972, at the Suceava Summer School organized by the Institute of Mathematics of the Academy of the Socialist Republic of Romania, in cooperation with the Society of Mathematical Sciences.

The study of the algebras of operators in Hilbert spaces was initiated by F. J. Murray and J. von Neumann, in connection with some problems of theoretical physics. The wealth of the mathematical facts contained in their fundamental papers interested many mathematicians. This soon led to the crystallization of a new branch of mathematics: the theory of algebras of operators. The first systematic exposition of this theory appeared in the well-known monograph by J. Dixmier [26], *which was subtitled* Algèbres de von Neumann. *It expounded almost all the significant results achieved until its appearance. Afterwards, the theory continued to develop, for it had important applications in the theory of group representations, in mathematical physics and in other branches of mathematics. Of great importance were the results obtained by M. Tomita, who exhibited canonical forms for arbitrary von Neumann algebras. In recent times fine classifications and structure theorems have been obtained for von Neumann algebras especially by A. Connes.*

The present book contains what we consider to be the fundamental part of the theory of von Neumann algebras. The book also contains the essential elements of the spectral theory in Hilbert spaces. The material is divided into ten chapters; besides the basic text, each chapter has two complementary sections: exercises, comments and bibliographical comments. The book ends with a bibliography, which includes all the titles we know of, which deal with the theory of algebras of operators and some related fields.

The reader is supposed to know only some elementary facts from functional analysis.

In writing this book we made use of existing books and courses (J. Dixmier [26], [42], *I. Kaplansky* [22], *J. R. Ringrose* [3], [4], [5], *S. Sakai* [10], [32], *M. Takesaki* [17], [18], *D. M. Topping* [8]*), as well as many articles, some of them available only as preprints. Some of the exercises are borrowed from J. Dixmier's book* [26]. *For the bibliography we made much use of Israel Halperin's Operator Algebras Newsletter.*

Thanks are due to Grigore Arsene and Dan Voiculescu for the help given during the writing of this book, for the useful discussions and for the bibliographical information they gave us.

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The Authors

Contents

Introduction

In the study of operator algebras there are two main methods, the first is of an algebraic character, while the second is more analytic.

The algebraic method proceeds by a successive reduction of problems concerning the arbitrary operators to problems about positive operators and from these to problems about projections, where one can avail oneself of the lattice-theoretical geometry of projections. In this geometry the main notion is that of equivalence and the main result is the comparison theorem, an important technical device being the polar decomposition of operators. These methods are elementary, but they afford a clear classification of the von Neumann algebras into general types. The results obtained by these methods are presented in Chapter 4 and in the first sections of Chapter 7.

The analytic method, which is more complex and profound, consists of a systematic manipulation with linear forms defined on operator algebras; they may be bounded, or unbounded. Here the important facts are concentrated around certain results which extend the classical Lebesgue-Radon-Nikodym theorem, the main technical tool here being the polar decomposition of linear forms. The analytic methods permit the analysis of relations existing between the given algebra and its commutant, as well as of those which relate the predual of the given algebra to the Hilbert space in which this algebra is operating. In Chapter 6 the relations existing between the type of the given algebra and of its commutant are studied, whereas Chapters 7 and 8 exhibit the quantitative relations which measure the relative wealth of the given algebra and of its commutant. For finite von Neumann algebras the existence of a trace which measures the relative dimension of projections allows the evaluation of the quantitative relations between the given algebra and its commutant by a coupling function of a metric nature. In other, more general, cases, the coupling between the given algebra and its commutant can be measured only by projective objects, namely cardinals associated with central projections, but the information thus obtained is not always satisfactory.

The von Neumann algebras which are well equilibrated with their commutants are called standard von Neumann algebras, and the main result of Chapter 10 is that any von Neumann algebra is isomorphic to a standard von Neumann algebra in a canonical form. This has been known for a long time in the case of the semifinite von Neumann algebras; to be extended to the general case, it required a new technique namely a "polar decomposition" for the involution of the algebra. Chapter 10 is dedicated to the study of the canonical forms of the von Neumann algebras as well as to some applications to the theory of arbitrary von Neumann algebras.

The theory of operator algebras is based on two fundamental results: the density theorem of J.von Neumann and the density theorem of I. Kaplansky, both presented in Chapter 3.

The present book covers results contained in M. Takesaki's work [18], and, with the exception of the reduction theory and of the examples of factors included there, those of J. Dixmier's book [26].

The reduction theory aims at decomposing an arbitrary von Neumann algebra into a family of von Neumann algebras with trivial centers (the so-called factors), in such a manner that the algebra be obtainable from this family, whereas its properties will be derivable from those of the factors. In this manner, the reduction theory transfers to the factors the purely non-commutative part of the algebra, whereas the commutative part is reflected in the space of the indices of the family of factors; the main problem of the structure and classification of the von Neumann algebras is thus reduced to the corresponding problems for factors. For the reduction theory one can read J. Dixmier's book [26], as well as the expository article by L. Zsidó [3], based on the ideas of S. Sakai [11]. Both develop the classical reduction theory of J. von Neumann, but from seemingly different points of view, which can easily be shown to be similar. For factor theory we recommend the works of J. Dixmier [26], [52], S. Sakai [32], D. McDuff [3], H. Araki and E. J. Woods [3], A. Connes [15], [19], [21 — 24]. Important results concerning the structure of von Neumann algebras are contained in the works of A. Connes [6], [7], and M. Takesaki [29], [33].

Our exposition refers to the spatial theory of von Neumann algebras, which considers them as being subalgebras of the algebra of all bounded linear operators on a Hilbert space. S. Sakai obtained in [3] the abstract characterization of von Neumann algebras and developed the theory of von Neumann algebras by non-spatial methods. Thus, in S. Sakai's book [32] the reader will find some of the results we present here, with different proofs. Also, S. Sakai's book [32] contains some other results which are not included in the present book.

"Algebras of operators" usually designate something more general than von Neumann algebras, the so-called C^*-algebras. In our exposition we have only incidentally referred to the C^*-algebras, but this theory makes full use of the theory of von Neumann algebras. For this theory, as well as for its applications to the theory of group representations, we refer the reader to J. Dixmier's monograph [42].

Other topics connected with the theory of operator algebras, but not treated in the present book, are the following: the problem of the generation of von Neumann algebras (see D. M. Topping [8], T. Saitô [10]), non-commutative harmonic analysis and duality theory for locally compact groups (see P. Eymard [1], M. Takesaki [23], M. Walter [2], [4], J. Ernest [5], [8]), non-commutative ergodic theory (see A. Guichardet [18]), applications to the theory of operators (see R. G. Douglas [3], [4], J. Ernest [9]), connections with some problems of theoretical physics (see D. Kastler [1], [3], G. E. Emch [2], D. Ruelle [4]).

Although rather a long time has elapsed since the publication of the works by F. J. Murray and J. von Neumann and their results are included in the books mentioned above, we consider that their works are still worth reading for those interested in the theory of operator algebras.

The present book is self-contained with complete proofs. The exercises contain results which enrich the text and which can be proved with the methods described in it; the more difficult exercises are marked by an asterisk, whereas some of the exercises which offer no difficulty are used in the main text and are marked by the symbol "!".

The final sections of each chapter include complements which contain bibliographical references, as well as the names of the mathematicians to whom the results contained in each chapter are to be ascribed.

The bibliography lists the works on operator algebras theory, as well as entries concerning the theory of group representations, mathematical physics and operator theory.

1

Topologies on spaces of operators

In this chapter we introduce the main topologies in the space $\mathscr{B}(\mathscr{H})$ of all bounded linear operators on a Hilbert space.

1.1. Lemma. Let $\mathscr{E}$ *be a vector space,* φ *a linear form on* $\mathscr{E}$ *and* $p_1, p_2, \ldots, p_n$ *semi-norms on* $\mathscr{E}$, *such that*

$$|\varphi(x)| \leqslant \sum_{k=1}^{n} p_k(x), \quad x \in \mathscr{E}.$$

Then there exist linear forms $\varphi_1, \ldots, \varphi_n$ *on* $\mathscr{E}$, *such that*

$$\varphi = \sum_{k=1}^{n} \varphi_k,$$

$$|\varphi_k(x)| \leqslant p_k(x), \quad x \in \mathscr{E}, \quad k = 1, \ldots, n.$$

Proof. Let $\mathscr{E}^n$ be the Cartesian product of n copies of $\mathscr{E}$, $\mathscr{D} \subset \mathscr{E}^n$ the diagonal of $\mathscr{E}^n$, p the semi-norm on $\mathscr{E}^n$ defined by

$$p(x_1, \ldots, x_n) = \sum_{k=1}^{n} p_k(x_k), \quad (x_1, \ldots, x_n) \in \mathscr{E}^n,$$

and $\tilde{\varphi}_0$ the linear form on $\mathscr{D}$ defined by

$$\tilde{\varphi}_0(x, \ldots, x) = \varphi(x), \quad x \in \mathscr{E}.$$

From the hypothesis we immediately infer that the linear form $\tilde{\varphi}_0$ on $\mathscr{D}$ is majorized on $\mathscr{D}$ by the semi-norm p. With the Hahn-Banach theorem we infer that there exists a linear form $\tilde{\varphi}$ on $\mathscr{E}^n$, having the following properties

$$\tilde{\varphi}(x, \ldots, x) = \tilde{\varphi}_0(x, \ldots, x), \; x \in \mathscr{E},$$

$$|\tilde{\varphi}(x_1, \ldots, x_n)| \leqslant p(x_1 \ldots, x_n), \quad (x_1, \ldots, x_n) \in \mathscr{E}^n.$$

We then define the forms φ_k by the relations

$$\varphi_k(x) = \varphi(0,\ldots,0,x,0,\ldots,0), \quad x \in \mathscr{E}, \quad k = 1,\ldots,n,$$

where, in the right-hand member, x stands on the k-th place.

The linear forms thus defined satisfy the conditions of the statement.

Q.E.D.

1.2. Let $\mathscr{E}$ be a Banach space, $\mathscr{E}^*$ the dual of $\mathscr{E}$ and $\mathscr{F}$ a vector subspace of $\mathscr{E}^*$. We denote by $\sigma(\mathscr{E};\mathscr{F})$ the weak topology defined in $\mathscr{E}$ by the family $\mathscr{F}$ of linear forms; then the $\sigma(\mathscr{E};\mathscr{F})$-topology is defined by the family of semi-norms $\{p_\varphi; \varphi \in \mathscr{F}\}$, where

$$p_\varphi(x) = |\varphi(x)|, \quad x \in \mathscr{E}.$$

We consider the norm topology in $\mathscr{E}^*$ and we denote by $\overline{\mathscr{F}}$ the closure of $\mathscr{F}$ in this topology. We denote by $\mathscr{E}_1$ the closed unit ball in $\mathscr{E}$.

Lemma. *Let $\mathscr{E}$ be a Banach space, $\mathscr{F} \subset \mathscr{E}^*$ a vector subspace and φ a linear form on $\mathscr{E}$.*

(i) *φ is $\sigma(\mathscr{E};\mathscr{F})$-continuous iff*) $\varphi \in \mathscr{F}$.*

(ii) *φ is $\sigma(\mathscr{E};\mathscr{F})$-continuous on $\mathscr{E}_1$ iff $\varphi \in \overline{\mathscr{F}}$.*

(iii) *The topologies $\sigma(\mathscr{E};\mathscr{F})$ and $\sigma(\mathscr{E};\overline{\mathscr{F}})$ coincide on $\mathscr{E}_1$.*

(iv) *If $\mathscr{F}$ is closed in the norm topology and φ is $\sigma(\mathscr{E};\mathscr{F})$-continuous on $\mathscr{E}_1$, then φ is $\sigma(\mathscr{E};\mathscr{F})$-continuous on $\mathscr{E}$.*

Proof. (i) Obviously, if $\varphi \in \mathscr{F}$, then φ is $\sigma(\mathscr{E};\mathscr{F})$-continuous. Conversely, if φ is $\sigma(\mathscr{E};\mathscr{F})$-continuous, then there exist $\psi_1,\ldots,\psi_n \in \mathscr{F}$, such that

$$|\varphi(x)| \leqslant \sum_{k=1}^{n} p_{\psi_k}(x), \quad x \in \mathscr{E}.$$

By virtue of Lemma 1.1, there exist linear forms $\varphi_1,\ldots,\varphi_n$ on $\mathscr{E}$, such that

$$\varphi = \sum_{k=1}^{n} \varphi_k$$

$$|\varphi_k(x), \leqslant p_{\psi_k}(x) = |\psi_k(x)|, \quad x \in \mathscr{E}, \quad k = 1,\ldots,n.$$

If $\psi_k = 0$, then $\varphi_k = 0$. If $\psi_k \neq 0$, then there exists $x_k \in \mathscr{E}$, such that $\psi_k(x_k) = 1$ and, for any $x \in \mathscr{E}$, we have

$$|\varphi_k(x - \psi_k(x)x_k)| \leqslant |\psi_k(x - \psi_k(x)x_k)| = 0.$$

Consequently, we have

$$\varphi_k = \varphi_k(x_k)\psi_k \in \mathscr{F} \text{ and } \varphi = \sum_{k=1}^{n} \varphi_k \in \mathscr{F}.$$

*) 'Iff' stands for 'if and only if'.

(ii) It is easily seen that if $\varphi \in \overline{\mathscr{F}}$, then the restriction of φ to $\mathscr{E}_1$ is $\sigma(\mathscr{E}; \mathscr{F})$-continuous. Conversely, let φ be a linear form on $\mathscr{E}$, whose restriction to $\mathscr{E}_1$ is $\sigma(\mathscr{E}; \mathscr{F})$ continuous. Then φ is norm-continuous and, therefore $\varphi \in \mathscr{E}^*$. Let $\varepsilon > 0$ be an arbitrary positive real number. Since the restriction of φ to $\mathscr{E}_1$ is $\sigma(\mathscr{E}; \mathscr{F})$-conitnuous at 0, we infer that there exist linear forms $\psi_1, \ldots, \psi_n \in \mathscr{F}$ such that:

$$\|x\| \leqslant 1, \ \sum_{k=1}^{n} p_{\psi_k}(x) < 1 \Rightarrow |\varphi(x)| < \varepsilon.$$

Hence we immediately infer that, for any $x \in \mathscr{E}$, we have

$$|\varphi(x)| \leqslant \varepsilon \|x\| + \|\varphi\| \sum_{k=1}^{n} p_{\psi_k}(x).$$

By virtue of Lemma 1.1, it follows that there exist linear forms φ_1, φ_2 on $\mathscr{E}$, such that

$$\varphi = \varphi_1 + \varphi_2,$$

$$|\varphi_1(x)| \leqslant \varepsilon\|x\|, \quad x \in \mathscr{E},$$

$$|\varphi_2(x)| \leqslant \|\varphi\| \sum_{k=1}^{n} p_{\psi_k}(x), \quad x \in \mathscr{E}.$$

Consequently, $\varphi_2 \in \mathscr{F}$ and $\|\varphi - \varphi_2\| = \|\varphi_1\| \leqslant \varepsilon$. Since $\varepsilon > 0$ was arbitrary, we get $\varphi \in \overline{\mathscr{F}}$.

Statements (iii), (iv) immediately follow from (i) and (ii).

Q.E.D.

1.3. Let $\mathscr{H}$ be a Hilbert space and $\mathscr{B}(\mathscr{H})$ the space of all bounded linear operators on $\mathscr{H}$. We consider $\mathscr{B}(\mathscr{H})$ as a Banach space only with respect to the usual operator norm:

$$\|x\| = \sup \{\|x\xi\|; \ \xi \in \mathscr{H}, \ \|\xi\| = 1\}.$$

For $\xi, \eta \in \mathscr{H}$ we define a linear form $\omega_{\xi,\eta}$ on $\mathscr{B}(\mathscr{H})$ by:

$$\omega_{\xi,\eta}(x) = (x\xi \mid \eta), \quad x \in \mathscr{B}(\mathscr{H}).$$

Obviously, $\omega_{\xi,\eta} \in \mathscr{B}(\mathscr{H})^*$ and it is easily checked that $\|\omega_{\xi,\eta}\| = \|\xi\| \cdot \|\eta\|$. The form $\omega_{\xi,\xi}$ will be simply denoted by ω_ξ.

Let $\mathscr{B}(\mathscr{H})_\sim$ be the vector space generated in $\mathscr{B}(\mathscr{H})^*$ by the forms $\omega_{\xi,\eta}$, $\xi, \eta \in \mathscr{H}$, whereas $\mathscr{B}(\mathscr{H})_*$ denotes the norm closure of $\mathscr{B}(\mathscr{H})_\sim$ in $\mathscr{B}(\mathscr{H})^*$.

Besides the *norm topology* we shall also consider the following topologies in $\mathscr{B}(\mathscr{H})$: the *weak operator topology*, or the *wo-topology*: it is the topology defined by the family of semi-norms

$$\mathscr{B}(\mathscr{H}) \ni x \mapsto |(x\xi \mid \eta)|, \quad \xi, \ \eta \in \mathscr{H};$$

in other words, it is just the $\sigma(\mathscr{B}(\mathscr{H}); \mathscr{B}(\mathscr{H})_\sim)$-topology;

the *strong operator topology*, or the *so-topology*: it is the topology defined by the family of semi-norms:

$$\mathscr{B}(\mathscr{H}) \ni x \mapsto \|x\xi\|, \quad \xi \in \mathscr{H};$$

the *ultraweak operator topology*, or the *w-topology*: it is, by definition, the $\sigma(\mathscr{B}(\mathscr{H}); \mathscr{B}(\mathscr{H})_*)$-topology.

We now apply Lemma 1.2, where we make $\mathscr{E} = \mathscr{B}(\mathscr{H})$, $\mathscr{F} = \mathscr{B}(\mathscr{H})_\sim$ and $\overline{\mathscr{F}} = \mathscr{B}(\mathscr{H})_*$, and by taking into account the terminology just introduced, we get the following

Lemma. *Let $\mathscr{H}$ be a Hilbert space. Then:*

(i) *$\mathscr{B}(\mathscr{H})_\sim$ is the set of all wo-continuous linear forms on $\mathscr{B}(\mathscr{H})$.*

(ii) *$\mathscr{B}(\mathscr{H})_*$ is the set of all w-continuous linear forms on $\mathscr{B}(\mathscr{H})$.*

(iii) *A linear form φ on $\mathscr{B}(\mathscr{H})$ is w-continuous iff its restriction to $\mathscr{B}(\mathscr{H})_1$ is wo-continuous.*

(iv) *In $\mathscr{B}(\mathscr{H})_1$ the wo-topology and the w-topology coincide.*

1.4. Theorem. *A linear form φ on $\mathscr{B}(\mathscr{H})$ is wo-continuous iff it is so-continuous.*

Proof. It is easy to see that the *so*-topology is finer (stronger), than the *wo*-topology; therefore, any *wo*-continuous linear form is *so*-continuous. Conversely, if φ is *so*-continuous, then there exist non-zero vectors $\xi_1, \dots, \xi_n \in \mathscr{H}$, such that

$$|\varphi(x)| \leqslant \sum_{k=1}^{n} \|x\xi_k\|, \quad x \in \mathscr{B}(\mathscr{H}).$$

From Lemma 1.1, there exist linear forms $\varphi_1, \dots, \varphi_n$ on $\mathscr{B}(\mathscr{H})$, such that

$$\varphi = \sum_{k=1}^{n} \varphi_k$$

$$|\varphi_k(x)| \leqslant \|x\xi_k\|, \quad x \in \mathscr{B}(\mathscr{H}), \quad k = 1, \dots, n.$$

Let $k \in \{1, \dots, n\}$ be any fixed index. We obviously have $\mathscr{H} = \{x\xi_k; x \in \mathscr{B}(\mathscr{H})\}$. As a consequence of what we have already proved, the mapping

$$x\xi_k \mapsto \varphi_k(x)$$

is a bounded linear form on $\mathscr{H}$. With Riesz' theorem we infer that there exists $\eta_k \in \mathscr{H}$, such that $\varphi_k(x) = (x\xi_k | \eta_k)$, $x \in \mathscr{B}(\mathscr{H})$.

Consequently, for any k there exists an $\eta_k \in \mathscr{H}$, such that $\varphi_k = \omega_{\xi_k, \eta_k}$. Therefore

$$\varphi = \sum_{k=1}^{n} \varphi_k \in \mathscr{B}(\mathscr{H})_\sim,$$

i.e., φ is *wo*-continuous.