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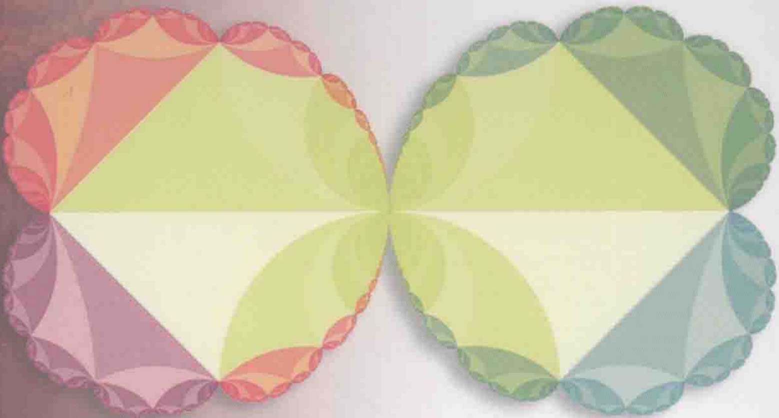

Editors

William Goldman
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Ser Peow Tan

GEOMETRY, TOPOLOGY AND DYNAMICS OF CHARACTER VARIETIES

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FOREWORD

The Institute for Mathematical Sciences at the National University of Singapore was established on 1 July 2000. Its mission is to foster mathematical research, both fundamental and multidisciplinary, particularly research that links mathematics to other disciplines, to nurture the growth of mathematical expertise among research scientists, to train talent for research in the mathematical sciences, and to serve as a platform for research interaction between the science community in Singapore and the wider international community.

The Institute organizes thematic programs which last from one month to six months. The theme or themes of a program will be chosen from areas at the forefront of current research in the mathematical sciences and their applications.

Generally, for each program there will be tutorial lectures followed by workshops at the research level. Notes on these lectures are usually made available to the participants for their immediate benefit during the program. The main objective of the Institute's Lecture Notes Series is to bring these lectures to a wider audience. Occasionally, the Series may also include the proceedings of workshops and expository lectures organized by the Institute.

The World Scientific Publishing Company has kindly agreed to publish the Lecture Notes Series. This Volume, "Geometry, Topology and Dynamics of Character Varieties", is the twenty-third of this Series. We hope that through the regular publication of these lecture notes the Institute will achieve, in part, its objective of promoting research in the mathematical sciences and their applications.

February 2012

Louis H.Y. Chen
Wing Keung To
Series Editors

PREFACE

This volume is based on a series of expository lectures presented at the highly successful graduate student summer school which kicked off the program *Geometry, Topology and Dynamics of Character Varieties* held at the National University of Singapore's Institute for Mathematical Sciences in July and August 2010. The theme was the character varieties of representations in a Lie group G of a discrete group Γ, the primary example being the case in which Γ is the fundamental group of a surface and G is $SL(2,\mathbb{R})$ or $SL(2,\mathbb{C})$. Character varieties lie at the confluence of many important areas of mathematics including algebraic geometry, hyperbolic geometry and Teichmüller theory, Kleinian groups and three-dimensional topology, dynamical systems and gauge theory. They have rich geometry and are related to interesting topological objects such as locally homogeneous geometric structures on manifolds and moduli spaces arising in gauge theory. Their study reveals many deep connections between these fields.

The summer school, which took place over three weeks with about 40 graduate students from over 12 countries participating, featured nine series of four lectures each, chosen so as to introduce important aspects of the background to the program. Following highly positive feedback, we felt it was important to preserve all this great material and enable it to be shared more widely. All but one of the lecturers were able to contribute, to which we have added two excellent expository articles based on lectures by students who attended the program (Koberda and Palesi). We hope that this resulting volume of edited and refereed articles will serve as a portal to a vibrant and many faceted area of mathematics.

Aimed at graduate students, much of the material in the volume is otherwise available only in specialized texts. Following Zhang's introduction to basic hyperbolic geometry in the synthetic axiomatic style, Aramayona and Leininger give an easily accessible introduction to the central topics

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of hyperbolic structures on surfaces and their degenerations via measured foliations and the Thurston compactification of Teichmüller space. Koberda explains the important technique of ping-pong lemmas, with applications in the context of hyperbolic geometry and mapping class groups of surfaces.

An important and unique feature of the summer school was Yamashita's specially designed lecture course which described how to write a program to draw limit sets and fundamental domains for Kleinian groups. Programming is done using the software python, and everything is explained from scratch without assuming any previous computing knowledge. Many people are interested in how to make the resulting beautiful pictures, and as far as we know, his is the only published article along these lines.

We then turn from the primary example of hyperbolic geometry to a more general context. Articles by Parker and Drumm provide introductions to complex hyperbolic and Lorentzian geometry, respectively. Palesi's article introduces a more general discussion about the representation space of surfaces groups into $SL(2,\mathbb{R})$ and its connected components, a topic taken further by Kim in the context of general Lie groups. The final article by Xia is an introduction to the topic of abelian and non-abelian cohomology which provides powerful analytical tools for the study of structures on the representation and character varieties.

The IMS program would not have been possible without contributions from many people. It was generously funded not only by NUS (from the IMS program funding as well as ARF grant R-146-000-133-112), but also aided by a grant from the NSF which enabled a substantial participation by US-based students and participants, and the Global Center of Excellence (Compview) of the Tokyo Institute of Technology which provided support for the Japanese contingent. We are grateful to the director of the IMS, Professor Louis Chen for his unwavering support and tremendous enthusiasm for the program, and all the staff at IMS, in particular, Emily, Claire, Stephen, Jolyn, Nurleen and Agnes for their help in running the program.

We would also like to thank the other scientific organizers of the program for their work and input: Javier Aramayona, Craig Hodgson, Sadayoshi Kojima, Yair Minsky, Makoto Sakuma, Jean-Marc Schlenker, Yan Loi Wong, Yasushi Yamashita and Ying Zhang. The contributions, of

Aramayona and Zhang in particular, to the smooth running of the program were invaluable.

December 2011

William Goldman
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CONTENTS

AN INVITATION TO ELEMENTARY HYPERBOLIC GEOMETRY

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We offer a short invitation to elementary hyperbolic plane geometry. We first examine the contents of Book I of Euclid's Elements and obtain a hyperbolic plane from the Euclidean one by negating Euclid's parallel postulate and keeping all of his other axioms. Then we explore the fundamentals of hyperbolic plane geometry, and study the structure of its isometries. Finally, we obtain certain identities involving the isometries and evaluate them in the upper half-plane model to derive some trigonometric laws for hyperbolic triangles.

Keywords: Non-Euclidean, hyperbolic plane, isometry, trigonometry

Mathematics Subject Classification 2000: 51M10

Introduction

In this short invitation to elementary hyperbolic geometry we choose to follow the approach of the discoverers of non-Euclidean geometry. Thus without giving any account of the history of the discovery of non-Euclidean geometry, we start by examining the contents of Book I of Euclid's Elements and obtain a hyperbolic plane from the Euclidean one by negating Euclid's parallel postulate and keeping all of his other axioms. We then explore the fundamentals of the hyperbolic plane geometry, study the structures of the isometries of a hyperbolic plane, and finally apply certain identities of isometries of the hyperbolic plane to derive trigonometric laws for triangles.

We choose this synthetic approach and as far as possible use no analytic models, because we believe this will provide the reader with more feeling for the geometry and thus enable him or her to explore the subject him or herself. In the more easily accepted analytic approaches, as this author has experienced, the reader has to rely on the chosen models to obtain any results and thus loses his or her precious geometric motivation. We hope that by studying the material presented in these notes, the reader will be able to develop the geometric ideas he/she has in mind analytically with no essential difficulties in any preferred model of a hyperbolic plane.

We have to warn the reader that we do not cover the trigonometry in detail and do not even touch the rich solid hyperbolic geometry. To master this omitted material, the reader is referred to some advanced textbooks as briefly discussed at the very end of the notes (in §4.6).

1. Euclid's Elements, Book I and Neutral Plane Geometry

1.1. *A brief review of contents of Elements, Book I*

In Book I of Elements, Euclid treated the fundamentals of the Euclidean plane geometry, including theories of triangles, parallels, and area. Precisely, Book I consists of 23 definitions, 5 postulates, 5 common notions, and 48 propositions. The definitions describe certain basic terms, of which we list only a few, such as point, line (curve), straight line, and surface, and then define some others based on them. The postulates are fundamental assumptions on the plane geometry while the common notions are commonly accepted assumptions on algebra or scientific reasoning. And, after that, the propositions (some are construction problems), including the famous Pythagorean Theorem (I.47), are presented in logical order. In what follows our phrasing of the contents of Book I of Elements is taken from Heath [13] or as appeared in Joyce's website http://aleph0.clarku.edu/~djoyce/java/elements/bookI/bookI.html".

Definitions (listed below are D1–D4 and D23):

D1. A point is that which has no part.

D2. A line is breadthless length.

D3. The ends of a line are points.

D4. A straight line is a line which lies evenly with the points on itself.

D23. (A pair of) parallel straight lines are straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction.

Postulates (P1–P5): Let the following be postulated.

P1. To draw a (finite) straight line from any point to any point.

P2. To produce a finite straight line continuously in a straight line.

P3. To describe a circle with any center and radius.

P4. That all right angles equal one another.

P5. That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, then the two straight lines, if produced indefinitely, meet on that side of the straight line on which are the two interior angles less than the two right angles.

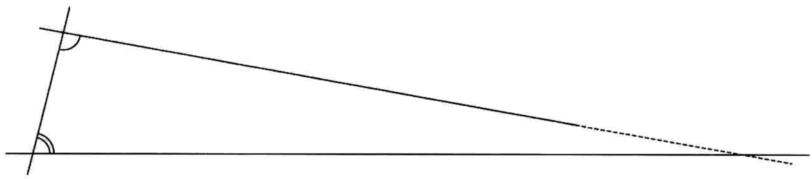

Fig. 1. Euclid's Postulate 5

Common Notions (CN1–CN5):

CN1. Things which equal the same thing also equal one another.

CN2. If equals are added to equals, then the wholes are equal.

CN3. If equals are subtracted from equals, then the remainders are equal.

CN4. Things which coincide with one another equal one another.

CN5. The whole is greater than the part.

Propositions (listed below are I.1–I.32 and I.47–I.48):

I.1. To construct an equilateral triangle on a given finite straight line.

I.2. To place a straight line equal to a given straight line with one end at a given point.

I.3. To cut off from the greater of two given unequal straight lines a straight line equal to the less.

I.4. If two triangles have the two sides equal to two sides respectively, and have the angles contained by the equal sides equal, then they also have the base equal to the base, the triangle equal to the triangle, and the remaining angles equal to the remaining angles respectively, namely those opposite the equal sides.

I.5. In isosceles triangles the angles at the base equal one another.

I.6. If in a triangle two angles equal one another, then the sides opposite the equal angles also equal one another.

I.7. Given two straight lines constructed from the ends of a straight line and meeting in a point, there cannot be constructed from the ends of

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the same straight line, and on the same side of it, two other straight lines meeting in another point and equal to the former two, namely each equal to that from the same end.

I.8. If two triangles have the two sides equal to two sides respectively, and also have the base equal to the base, then they also have the angles equal which are contained by the equal straight lines.

I.9. To bisect a given rectilinear angle.

I.10. To bisect a given finite straight line.

I.11. To draw a straight line at right angles to a given straight line from a given point on it.

I.12. To draw a straight line perpendicular to a given infinite straight line from a given point not on it.

I.13. If a straight line stands on a straight line, then it makes either two right angles or angles whose sum equals two right angles.

I.14. If with any straight line, and at a point on it, two straight lines not lying on the same side make the sum of the adjacent angles equal to two right angles, then the two straight lines are in a straight line with one another.

I.15. If two straight lines cut one another, then they make the vertical angles equal to one another.

I.16. In any triangle the exterior angle is greater than either of the interior and opposite angles.

I.17. In any triangle the sum of any two angles is less than two right angles.

I.18. In any triangle the angle opposite the greater side is greater.

I.19. In any triangle the side opposite the greater angle is greater.

I.20. In any triangle the sum of any two sides is greater than the remaining one.

I.21. If from the ends of one of the sides of a triangle two straight lines are constructed meeting within the triangle, then the sum of the straight lines so constructed is less than the sum of the remaining two sides of the triangle, but the constructed straight lines contain a greater angle than the angle contained by the remaining two sides.

I.22. To construct a triangle out of three straight lines which equal three given straight lines: thus it is necessary that the sum of any two of the straight lines should be greater than the remaining one.

I.23. To construct a rectilinear angle equal to a given rectilinear angle on a given straight line and at a point on it.

I.24. If two triangles have two sides equal to two sides respectively, but have one of the angles contained by the equal straight lines greater than the other, then they also have the base greater than the base.

I.25. If two triangles have two sides equal to two sides respectively, but have the base greater than the base, then they also have the one of the angles contained by the equal straight lines greater than the other.

I.26. If two triangles have two angles equal to two angles respectively, and one side equal to one side, namely, either the side adjoining the equal angles, or that opposite one of the equal angles, then the remaining sides equal the remaining sides and the remaining angle equals the remaining angle.

I.27. If a straight line falling on two straight lines makes the alternate angles equal to one another, then the two straight lines are parallel to one another.

I.28. If a straight line falling on two straight lines makes the exterior angle equal to the interior and opposite angle on the same side, or, equivalently, the sum of the interior angles on the same side equal to two right angles, then the two straight lines are parallel to one another.

I.29. A straight line falling on two parallel straight lines makes the alternate angles equal to one another, the exterior angle equal to the interior and opposite angle, and the sum of the interior angles on the same side equal to two right angles.

I.30. Straight lines parallel to the same straight line are also parallel to one another.

I.31. To draw a straight line through a given point parallel to a given straight line.

I.32. In any triangle, if one of the sides is produced, then the exterior angle equals the sum of the two interior and opposite angles, and the sum of the three interior angles of the triangle equals two right angles.

I.47. In right-angled triangles the square on the side opposite the right angle equals the sum of the squares on the sides containing the right angle.

I.48. If in a triangle the square on one of the sides equals the sum of the squares on the remaining two sides, then the angle contained by the remaining two sides is right.

1.2. *A useful lemma*

The following useful lemma, called the Bow Tie Lemma by some authors, is an easy consequence of the SAS congruence criterion.

Lemma 1.1. *Given $\triangle ABC$, let M be the midpoint of side BC. Produce*

AM to D so that $|AM| = |MD|$*. Then* $\triangle ACM \cong \triangle DBM$*.*

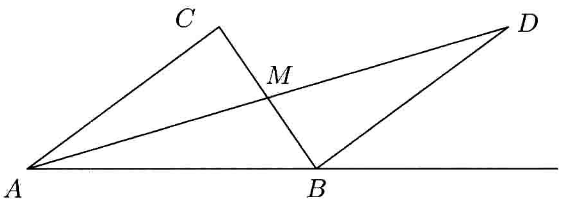

Fig. 2. Figure for Lemma 1.1

Euclid used Lemma 1.1 in Elements to give a clean proof of Proposition I.16, an important proposition in Book I. (We leave it to the reader to try to find a not-so-clean but easier proof for Proposition I.16.)

1.3. *A figure-free proof of Proposition I.7*

Proposition I.7 is a stronger version of I.8, the SSS congruence criterion. The proof that Euclid gave relies, however, on how the figure is drawn, and omits a case. Here we present a complete, figure-free proof.

Let $\triangle ABC$ and $\triangle ABD$ be two triangles such that C and D are distinct points on the same side of the straight line AB, and such that $|AC| = |AD|$ and $|BC| = |BD|$. We proceed to deriving a contradiction. Let M be the midpoint of CD. Then M lies on the same side of AB as C and D. By Propositions I.5 and I.4, we conclude that straight lines AM and BM are both perpendicular to CD and therefore coincide. Hence M lies on straight line AB, which is absurd. This proves Proposition I.7.

1.4. *More notes on Elements, Book I*

In our discussion of plane geometry, we do not pursue a pure axiomatic way and assume that everything under discussion occurs in the same plane which is, topologically, the usual plane.

Taken as granted, Euclid assumed that each finite straight line has its definite magnitude, *length*, which is additive when two finite straight lines lying in the same straight line are juxtaposed. He also assumed that an angle has its definite measure which is also additive when two angles with the same vertex are juxtaposed.

For the angles, Euclid further required in Postulate 4 that the full round angles (four right angles) at all points in the plane equal one another.

The definition, D4, of straight line is a vague description since one cannot conclude a line (curve) be a straight line merely by this definition.

Postulate 1, according to the way Euclid used it, should be interpreted as: Given two distinct points in the plane, there exists in the plane one and *only one* straight line which connects them. Therefore, if two distinct straight lines in the plane intersect, they intersect in only one point. Consequently, we conclude that each interior angle of a triangle is less than two right angles. Furthermore, as a consequence of Postulate 1 and Proposition I.4, we conclude that the perpendicular straight line drawn from a given point to a given straight line is unique (note that the existence of the perpendicular is guaranteed by Propositions I.14 and I.15).

We notice that Euclid's system of axioms for the plane geometry is incomplete. For example, Proposition I.4, the SAS congruence criterion, should be regarded as an axiom instead of a theorem; in fact, Euclid's proof for I.4 is not satisfactory. In so doing, Proposition I.5 follows as an easy consequence of (axiom) I.4. One also needs to include a continuity axiom to confirm that if a straight line enters the interior of a triangle then it will leave the region when the straight line is indefinitely produced in that direction. It is well known that a complete system of axioms for the Euclidean plane geometry was given by Hilbert in [14].

Proposition I.20 is the so-called triangle inequality.

Proposition I.21 can be rewritten as: If point D lies within triangle ABC, then $|AD| + |DB| < |AC| + |CB|$ and $\measuredangle ADB > \measuredangle ACB$.

Proposition I.26 establishes the ASA and AAS congruence criteria.

Proposition I.27 gives, without making use of Postulate 5 in its proof, parallel straight lines: If one straight line falling on two straight lines makes the alternate interior angles equal to one another, then the two straight lines are parallel. This can be easily proved using the SAS congruence criterion and the uniqueness of intersection points of two straight lines. Hence I.27 can be put just after I.4 if one wishes.

A careful reader will notice that Euclid did not use his Postulate 5 until in the proof of Proposition I.29, the Euclidean Parallel Theorem, which asserts that whenever one straight line falls on two parallel straight lines, the alternate angles equal one another. In other words, each of the first 28 propositions in Elements, Book I can be proved using only the axioms (including the hidden ones to be added in) of the plane geometry other than Euclid's Postulate 5. This leads to the term "neutral plane geometry" which we shall discuss shortly.

Of the very last two propositions in Book I, I.47 is the famous

Pythagorean Theorem, and I.48 is the converse of I.47. The proofs of them, of course, essentially make use of Postulate 5, at least in the way Euclid had proved them.

1.5. *Playfair's axiom*

One would wonder why Euclid made his Postulate 5 so complicated a statement compared with the other four, and, in particular, Euclid even did not mention parallels in it. This is partly because the ancient Greek philosophy avoids any unnecessary use of infinity which lies in the nature of parallels. For the same reason Euclid never used the term "infinite straight line".

On the other hand, with the presence of all axioms of the Euclidean plane geometry other than Euclid's Postulate 5, it is easy to show that Postulate 5 is equivalent to the so-called Playfair's Axiom below.

Playfair's axiom. Through a given point P not on a given straight line l, there passes at most one straight line which is parallel to l.

Since, by I.27, there exists at least one parallel, the phrase "at most one" in Playfair's Axiom can be replaced, if one wishes, by "exactly one".

1.6. *Neutral plane geometry*

By neutral geometry, for which J. Bolyai used the term "absolute geometry", we mean the geometry obtained from the Euclidean geometry by dropping just Euclid's Postulate 5.

In particular, all the first 28 propositions in Euclid's Elements, Book I are indeed theorems (possibly with I.4 chosen as an axiom) of neutral plane geometry, since their proofs make no use of Postulate 5.

1.7. *Angle-sums of triangles and Legendre's Theorems*

We have seen that, with the presence of all other axioms for the plane geometry, Euclid's Postulate 5 is equivalent to Playfair's Axiom. It is not hard to see that they are also equivalent to the Euclidean Angle-Sum Axiom (Proposition I.32).

Euclidean angle-sum axiom. The sum of the interior angles of every triangle equals two right angles.

Regarding angle-sums of triangles in a neutral plane, we have the well-known theorems of Saccheri and Legendre.

Theorem 1.2 (Legendre's First Theorem). *In a neutral plane the sum of the angles of a triangle is less than or equal to two right angles.*

Proof. Suppose on the contrary that there exists a triangle, $\triangle ABC$, with angle-sum greater than two right angles. With no loss of generality, we may assume that the angle at A is the smallest angle of $\triangle ABC$. Then by Lemma 1.1, we obtain a triangle $\triangle ABD$ with the same angle-sum as $\triangle ABC$, such that the smallest angle of $\triangle ABD$ is at most half the smallest angle of $\triangle ABC$. By Archimedes's Axiom, after a finite number of steps one arrives at a triangle with the same angle-sum as $\triangle ABC$ and with smallest angle less than the excess of its angle-sum over two right angles. Then this triangle has sum of certain two angles greater than two right angles, contradicting Proposition I.17. This proves Theorem 1.7. □

By virtue of Legendre's First Theorem, we may define the defect of a triangle in a neutral plane to be the deficiency of its angle-sum to two right angles. Similarly, we define the defect of a (simple) quadrilateral in a neutral plane to be the deficiency of its angle-sum to four right angles.

Definition 1.3. The *defect* $\delta(\triangle ABC)$ of triangle ABC equals two right angles minus the sum of the angles of $\triangle ABC$ and is therefore nonnegative. Similarly, the *defect* $\delta(\square ABCD)$ of (simple) quadrilateral $ABCD$ equals four right angles minus the sum of the interior angles of $\square ABCD$ and is also nonnegative.

The defect so defined is additive under subdivision of a triangle or quadrilateral into smaller triangles and quadrilaterals. Below we list two simple cases.

Proposition 1.4. *In triangle ABC let D be a point within side BC. Then*

$$\delta(\triangle ABC) = \delta(\triangle ABD) + \delta(\triangle ADC).$$

Proposition 1.5. *If quadrilateral $ABCD$ is obtained as the union of triangles ABC and ACD which share no common interior points, then*

$$\delta(\square ABCD) = \delta(\triangle ABC) + \delta(\triangle ACD).$$

The following theorem establishes the universality of the Euclidean Angle-Sum Axiom, namely, if the axiom is satisfied by one triangle in the plane then it is satisfied by every triangle.

Theorem 1.6 (Legendre's Second Theorem). *In a neutral plane if one triangle has angle-sum equal to two right angles, then so does every triangle.*

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