

Mark Millonas
Editor



Fluctuations and Order

The New Synthesis



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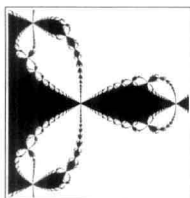
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Preface

The volume that you have before you is the result of a growing realization that fluctuations in nonequilibrium systems play a much more important role than was first believed.¹ It has become clear that in nonequilibrium systems noise plays an active, one might even say a creative, role in processes involving self-organization, pattern formation, and coherence, as well as in biological information processing, energy transduction, and functionality. Now is not the time for a comprehensive summary of these new ideas, and I am certainly not the person to attempt such a thing. Rather, this short introductory essay (and the book as a whole) is an attempt to describe where we are at present and how the viewpoint that has evolved in the last decade or so differs from those of past decades.

Fluctuations arise either because of the coupling of a particular system to an external unknown or “unknowable” system or because the particular description we are using is only a coarse-grained description which on some level is an approximation. We describe the unpredictable and random deviations from our deterministic equations of motion as noise or fluctuations. A nonequilibrium system is one in which there is a net flow of energy.

There are, as I see it, four basic levels of sophistication, or paradigms, concerning fluctuations in nature. At the lowest level of sophistication, there is an implicit assumption that noise is negligible: the *deterministic paradigm*. This is the assumption that is always made whenever one studies deterministic models of natural processes.

A great deal of progress has been made in the study of deterministic dynamics systems, and in our own time the study of nonlinear dynamics and chaos has radically transformed the way we look at such systems. Twenty years ago, scientists found it hard to believe that deterministic systems could exhibit low-dimensional chaotic behavior that was indistinguishable from random motion. Today it seems that these ideas often so dominate the intellectual climate that many scientists have a hard time believing there is any other type of randomness. It is currently quite popular when one is faced with a system that exhibits unpredictable behavior to apply tools appropriate to the analysis of chaotic deterministic systems, even when these systems are often demonstrably random at a fundamental level. Such an approach is even less valid now that it is beginning to be understood that the addition of even small amounts of noise to a nonlinear dynamical system can alter its statistical behavior in a fundamental way—a change that has nothing to do with the “sensitive dependence on initial conditions” in chaotic systems.

¹The title of this book is a homage to E.O. Wilson’s integrative work, *Sociobiology: The New Synthesis*.

The use of deterministic systems as models of nature depends on the implicit assumption that nonlinear, nonequilibrium systems can be separated into a deterministic part that contains the essence, and a random part that can be thrown away. Often this viewpoint is very successful. However, just as it is not possible to separate nonlinear dynamical systems into simple parts that can be understood separately, it is not always possible to separate the deterministic element from the stochastic element in systems subject to fluctuations. Since randomness enters into the fundamental processes that make up many systems in nature, it is probably impossible to understand the vast majority of these processes, even qualitatively, without explicitly incorporating this randomness into our models.

On the next higher level, noise is often regarded as a source of pure disorder: the *equilibrium paradigm*. This is the sometimes misleading lesson of equilibrium statistical mechanics. In this picture, noise is included, but still a deterministic dynamics controls everything. The noise merely provides for fluctuation about the deterministic, stable limit sets, disrupting orderly or coherent behaviors as it is increased. Many researchers organize their understanding of a particular complex system around such pictures without any clear understanding of their validity (and sometimes even without conscious awareness).

On a still more sophisticated level is the concept of “order through fluctuations,” put forward by Prigogine and others, according to which the amplification of fluctuations near an instability leads to a more ordered macroscopic state. At the risk of oversimplification, I will call this idea the *passive noise paradigm*, in that only the transitions to certain ordered macroscopic states and not the states themselves are influenced by the fluctuations. In using this paradigm, one often assumes that the probability that a system will be found in one of a number of possible ordered states after the transition can be determined from macroscopic criteria, without reference to the detailed kinetics.

While the passive noise paradigm often applies to systems composed of a great number of parts, the accompanying assumption is in general not valid (see chapter 1). In systems where the internal fluctuations are large, or where there are external fluctuations, neither the paradigm nor the assumption applies; however, this is the category into which many important nonequilibrium systems in nature fall.

Finally, there is the center around which much of this book is focused, which I will call the *active noise paradigm*. In general, even when the fluctuations are small, the probability of a macroscopic state depends on the explicit details of the global kinetics and cannot be determined from the macroscopic state alone. In addition, if the fluctuations are macroscopically large, as is the case in many complex systems, the qualitative structure of the macroscopic states, as well as their relative probabilities, will also depend on the global kinetics. In simple terms, this means that a knowledge of the global kinetics is necessary to understand nonequilibrium systems, even near the stable points.

In such cases, one cannot eliminate the fluctuations from consideration because they also contain important qualitative information about the behavior of the system. Such systems cannot be described by deterministic or passive noise because there are no macroscopic thermodynamic variables capable of describing the sys-

tem. Put another way, important nonequilibrium effects are inextricably mixed in both the macroscopic net forces and the fluctuating forces.

The cases to which active noise applies are to be contrasted strongly with those described by passive noise. In fact, such cases represent an ever growing, and at this point definitive, body of evidence against the general applicability of the ideas of the Prigogine school. Complex is complex, and there are no magic prescriptions, as Rolf Landauer points out in the first chapter. If it is true that many of our present intuitive notions are not entirely valid ways of understanding nature, we may need to radically reappraise the role that noise plays in the behavior of such systems. This book is an attempt to begin such a reappraisal.

A word needs to be said about the topics represented in this book. It goes without saying that they represent my own unique and perhaps warped view, along with the views of the editors of this series. I have chosen to diminish the representation of important subjects (from the pool of submitted papers) such as self-organized criticality and stochastic resonance because entire conferences are now devoted to these subjects.

It is my main goal in this book to illustrate clearly the wide intellectual scope of the subject rather than attempt to subsume all under one or another all-encompassing rubric. I also wanted to show that a great variety of researchers, from theoretical cosmologists to experimental biologists, are contributing to the subject. If some of the chapters in this book are a bit speculative, or a bit outside the strict epicenter of our subject, so be it. The generation of new ideas is vital for the health of any subject, for without them we slip into the stagnant waters.

Many people have contributed to the publication of this volume. It has been a long road but, judging by the final result, I think it has been worth it. Thanks are due to the powers that be at CNLS and the Theoretical Division at the Los Alamos National Laboratory for providing funds and experience for the original workshop at Los Alamos in 1993. In particular, I thank Alan Bishop, Don Cohen, Gary Doolen, Mac Hyman, and Alan Lapedes. The Santa Fe Institute provided some additional financial support for the meeting. Without the able organizational skills and experience of Barbara Rhodes, chaos would have reigned at the conference. Katja Lindenberg made it possible for this volume to be published as part of the INLS series. I would particularly like to thank Tracy Lopez and Elizabeth Henry for helping me with the manuscript in their spare time. Without them, I would still be hunting and pecking my way through the papers—which reminds me to thank those authors who turned in their contributions in the requested form and didn't make us retype the whole thing. Lastly, I acknowledge the patience and professionalism of the people at Springer-Verlag.

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