modern mathematics
an elementary approach

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Brooks/Cole Publishing Company,
Belmont, California

A Division of Wadsworth Publishing
Company, Inc.

Dedicated to Joyce, Eddy, Paul

Brooks/Cole Publishing Company, Belmont, California. A Division of Wadsworth Publishing Company, Inc.

L.C. Cat. Card No.: 66–15873
Printed in the United States of America

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preface

This text is designed to provide the mathematical concepts needed to teach any of the so-called "new approaches" found in modern arithmetic textbooks and to prepare prospective teachers for inevitable future changes in mathematics curricula. It should also be interesting and valuable to the liberal arts student who desires an appreciation and understanding of the basic structure of mathematics. It is also appropriate for in-service courses and institute training of experienced teachers at both the elementary and the junior high school levels. It grew out of material accumulated in the process of teaching prospective elementary school teachers, teaching in (and directing) one of the first National Science Foundation Summer Institutes, and participating in a summer conference on the training of elementary school teachers, which was held at the University of Arizona in 1963.

The CUPM recommends that all prospective elementary school teachers have two semesters of study of the structure of the number system, one semester of algebra, and one semester of geometry. This book contains most of the concepts recommended for these four courses, although it is designed for a six semester-hour course.

No discussion of the structure of number systems could be complete without some attention to the nature of mathematical proofs. Consequently, this text develops a number of mathematical proofs. Each theorem is supported by intuitive discussion and illustrated by examples.

The language and style throughout the book will appeal to students with diverse mathematical backgrounds. Two years of use in a formal college course have demonstrated that the book's simplicity of language, intuitive discussion, and numerous examples tend to facilitate study and understanding.

The problems in the exercise sets vary from very easy to very difficult. The more difficult problems are indicated by asterisks and may be omitted at the discretion of the teacher. Many of the problems that involve proofs are so marked.

Since one objective of the book is to encourage the student to appreciate the many years of work necessary for the evolution of our number system, historical ideas have been incorporated throughout rather than relegated to a separate chapter.

Chapter 1 presents a verbal approach to the logic of mathematics, including definitions and terminology. The concept of a mathematical system and a brief discussion of deductive reasoning serve as stepping stones to the remainder of the book. Chapter 2 introduces the language of sets and relations, which is used throughout the book. Chapters 3 through 7 present a step-by-step development from the concept of natural numbers to the real-number system, including a complete discussion of number bases in Chapter 4. Useful properties of numbers are examined in Chapter 8. In Chapter 9, modular arithmetic is used to develop the concept of "casting out 9's and 11's" and to illustrate such mathematical systems as groups and fields. Chapters 10 and 11 present a rather complete discussion of elementary geometry, and Chapter 12 briefly summarizes algebraic concepts not covered in other sections.

The first six chapters and the last six chapters contain 45 lessons per group; the book is thus easily divided into two equal consecutive semesters, each covering six chapters. If the entire book is to be covered in one semester, the following sequence is suggested: Sections 1, 5, and 6 of Chapter 1; Chapters 2, 3, and 4; Chapters 5 and 6, without the developmental material; Sections 1 through 4 of Chapter 7; and Sections 1 through 5 of Chapter 8. This elimination of material also provides 45 lessons, and should prove cohesive.

Sections that are not essential for the study of the remainder of the book are indicated by asterisks preceding the section number.

I would like to express my appreciation to President Leslie S. Wright and Dean John A. Fincher of Samford University, who granted the sabbatical leave during which this book was written. In addition, I am indebted to the many students and elementary school teachers who studied this book in the form of a mimeographed manuscript. Particular gratitude goes to Dr. Robert J. Wisner, former Executive Director of the Committee on the Undergraduate Program in Mathematics, who contributed valuable suggestions. Professors Irving Drooyan of Los Angeles Pierce College, Daniel J. Ewy of Fresno State College, F. J. Lorenzen of the University of California at Davis, Robert L. Poe of Kansas State Teachers College, and Karl Stromberg of the University of Oregon all read early versions of the manuscript and contributed valuable criticisms. Appreciation is also owed to Dr. William D. Peeples, Mr. Sanders Bishop, Mr. Joe Faulkner, Mr. Richard Morris, Miss Anita Howard, and Mrs. Robert Yeager, who used this book in mimeographed form and made many suggestions. Finally, I would like to thank the graduate assistants, student assistants, and secretaries of the Mathematics Department of Samford University for their patience and understanding.

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1 introduction to mathematical reasoning

1 mathematics is many things to many people

Approach your college friends with this question: "What is mathematics?" Ask your faculty adviser, one of your teachers. Ask a neighbor, your parents. Ask *yourself* the question "What is mathematics?" Compare the answers. Are you surprised at the varied responses?

Some will say that mathematics is working with numbers in answering the questions "how many?" and "how much?" In this role mathematics is a *tool*, a collection of skills which may be used for calculation and problem solving. Thus mathematics as a tool becomes indispensable to our modern world of business transactions, industrial production, and scientific research. In a similar manner, mathematics assumes importance for you. As a competent citizen you must be able to cope with taxes, interest, budgets, grocery bills, and so on, all of which depend on mathematics. Yet mathematics is more than this.

To some of your friends mathematics may be a *science*. This assumption is again correct. In the sense of its precision, in its rigor of development, in its search for truth, mathematics is the ultimate science, a science of logical reasoning. A study of mathematics involves a study of methods for drawing conclusions from assumed premises. From some of the greatest minds of every age have come contributions to the systematized mathematical knowledge that is in use today.

Some compare mathematics to a *game*. In any game certain rules and regulations must be learned in order to play. If at any time a rule is broken or ignored, the game ceases to be fair. The same is true in mathematics. One begins with given *rules* or *laws* (sometimes called *postulates*) and plays the game of logical reasoning. If at any time a law is broken, the mathematical reasoning is no longer valid. One notes that, just as the rules for a game vary depending on the game being played, the given laws and definitions in our mathematical reasoning vary depending on the type of work involved. Whether the laws describe any physical or geometric situation within the realm of experience has nothing whatsoever to do with the process of reasoning. If the conclusions are based on laws and definitions, and if logical reasoning has been used, then the mathematician is satisfied.

The reply of a research mathematician to the question "What is mathematics?" may have been "Mathematics is an *art*." Certainly there is aesthetic satisfaction in the development of new mathematical theories, in the construction of new systems, new concepts, new ideas. It is difficult to describe creative activity or to specify what kind of reasoning mathematics is. However, new mathematical concepts are being developed daily; indeed, man has developed more mathematics since 1900 than in all previous time. These facts indicate that mathematics may be something that man has created rather than something he has discovered.

Did anyone classify mathematics as a *language*? For most people, whether they realize it or not, mathematics is most definitely a language for expressing their ideas. In order to compare the weight of two football players, the average grades of students in a class, and innumerable other subjects, one must employ the language of mathematics. Just as the artist expresses certain feelings and thoughts through the media of painting and sculpturing, one uses mathematics to express ideas of quantity and order. The language of mathematics is needed to converse fluently, to express relationships, to make comparisons, to quote statistics, to reach conclusions. Human beings are expected daily to express themselves creatively in all fields of endeavor. Yet many people cannot communicate certain ideas because they cannot read or write the language of mathematics. To have a well-rounded education one must be able to express ideas, plans, and beliefs clearly; this is difficult to accomplish without a knowledge of modern mathematics.

Now are you thoroughly confused? You have heard that mathematics is a tool, an art, a knowledge, a science, and a language. Thus you see that mathematics means different things to different people. However, the important thing to remember is that mathematics enters everyone's life in some way.

Consequently, this textbook will attempt to present a clear understanding of some modern mathematics. Efforts will be made to demonstrate clearly how mathematical systems are formed; and through this approach to the study of mathematics, you will be able to retain and use for a longer period of time the mathematics that you learn.

The study of this material should not only furnish the information needed for the practical daily use of mathematics but also provide patterns and procedures for solving some everyday problems of life. You should acquire the ability to study and analyze situations, to think logically, to argue deductively. It is hoped that ten years from now you will be able to say that mathematics was one of the most rewarding parts of your college curriculum.

*2 inductive reasoning

The following three sections will outline the processes by which logical conclusions are obtained. Reasoning, or drawing conclusions, can be classified into two categories—*inductive reasoning* and *deductive reasoning.* When a person makes observations and on the basis of these observations arrives at a conclusion, he is reasoning *inductively.* A small child feels the heat coming from a stove and after a few observations concludes that the stove is hot. Arriving at a conclusion on the basis of repeated scientific experimentation is sometimes called empirical inference or inductive inference, but it still involves inductive reasoning. A statistician collects and organizes his observations and then uses the information obtained to reach conclusions. Similarly, a chemist performs the same experiment many times under the same initial conditions. When he obtains the same result each time, he is convinced that this result will be obtained each time he performs the experiment. He is reasoning inductively: *arriving at a general conclusion from particular observations.*

This type of reasoning is not new in mathematics. Prior to the classical Greek civilization, mathematics was largely inductive in nature. Most mathematical formulas were "rule of thumb" procedures which were found to be approximately correct. Many were obtained by trial and error processes and later proved incorrect. For instance, at one time the circumference of a circle was thought to be three times the diameter. An Egyptian formula used the square of 8/9 of a circle's diameter as the area of a circle. We know now that both conclusions were wrong, and the formulas we use for these measurements have been proved to be true by methods of thought not involving physical measurements.

One makes repeated use of inductive reasoning in everyday affairs. You have heard the statement, "Experience is the best teacher." For example, on a cloudy day you carry your umbrella because you have noticed a relationship between clouds and rain. However, inductive reasoning is not always as simple as this example indicates. To be good at inductive reasoning, one must train himself to notice the important elements in a situation and to understand these elements. A doctor often uses inductive reasoning when he examines a patient and makes a diagnosis. The doctor solves the problem, but the untrained person will fail because he does not recognize or understand the important facts of his observation.

It is unfortunate that many people believe that mathematics uses deductive reasoning exclusively. This is due in part to the fact that most *published* mathematics involves deductive reasoning. However, for every page of published mathematics, dozens of pages of unsuccessful attempts have been discarded. Inductive reasoning is essential to the creation of new mathematics. The development of new mathematics usually starts with conjectures. A *conjecture* is a statement which is thought, usually with good reason, to be true but which has yet to be proved true. Months or years often elapse before some conjectures are proved or disproved. Some conjectures remain unproved even after hundreds of years.

A very famous unsolved problem is the Goldbach conjecture for prime numbers. (A counting number other than 1 is said to be a *prime* if it is divisible only by itself and 1.) In 1742 Goldbach stated his conjecture that every even number greater than 4 is the sum of two odd primes. For example, 8 is the sum of 3 and 5, 10 is the sum of 3 and 7, 12 is the sum of 5 and 7, and so on. He never found a proof for this conjecture, and, despite efforts of outstanding mathematicians, neither has anyone else.

Consider the conjecture that the formula $N^2 - N + 41$ will give only prime numbers if one substitutes counting numbers for N. If you substitute 1 for N, you will get 41, which is a prime; if you substitute 2, you get 43; if you substitute 3, you get 47. Continue this process by substituting 4, 5, 6 to obtain 53, 61, and 71, respectively. Now 41, 43, 47, 53, 61, and 71 are all primes. Does it follow that if any counting number is substituted for N a prime is produced? Is the conjecture correct? (Try substituting 41 for N.)

It is hoped that this discussion will emphasize the fact that it is good to make conjectures—to reason inductively—but conclusions drawn from this type of reasoning may not be correct. In mathematics one must prove conjectures by processes called deductive reasoning, which will be discussed in the next section.

exercise set 1-1

1. Explain in a short paragraph
   (a) in what sense mathematics is an invention and not a discovery.
   (b) the inductive method of reasoning.
2. Give three examples of inductive reasoning.
3. Use the mathematics library to find grounds for calling mathematics
   (a) a tool. (b) an art. (c) a language.
4. Can you answer the following questions?
   (a) A clock strikes three in 2 seconds. How long does it take to strike nine?
   (b) A rectangular house is so built that every wall has a window opening on the south. A bear is seen from one of the windows. What color is the bear?
   (c) Is it legal for a man to marry his widow's sister?

*3 propositions or sentences

Even though we will not attempt in this book to give a complete discussion of symbolic logic, we do need some of the vocabulary associated with this area of study.

definition 1-1. *A sentence or proposition is an assertion assigned a truth value; that is, it can be meaningfully classified as either true or false.*

"The temperature yesterday averaged 30 degrees," "President Lincoln was born in Illinois," "The wind is blowing," "2 + 3 = 5," and "8 is less than 5" are examples of propositions. Notice that all these statements assert something and can be classified as either true or false. We will often use letters such as p, q, r,... to represent propositions.

"Good-morning," "Woodman, spare that tree," "Do not stare at me," and "Keep your mind open," are not propositions. None of these statements has a truth value. "He is president of the Owl's Club" and "$x + 4 = 9$" are propositions if we know whom *He* refers to and what number x represents. Otherwise they are not propositions.

definition 1-2. *The negation of proposition p is the proposition "It is not true that p."*

Equivalently, the negation of a proposition is the expression "It is false that," followed by the proposition itself. The negation of "New York City is the capital of the United States" is "It is not true that New York City is the capital of the United States." The use of the grammatical form "It is not true that p" sometimes results in an awkward sentence structure. For many simple propositions, the negation may be more simply expressed by negating the predicate. For example, the negation of the preceding proposition could be written as "New York City is not the capital of the United States." The negation of a proposition is defined to be false when the proposition is true, and true when the proposition is false.

Sometimes it is difficult to state the negations of propositions involving the word "all" or the word "some." In mathematics the word "some" always means *at least one and perhaps all.* On the other hand, "all" means *every one.* The negation of the proposition, "All men have black hair," can be written, "It is not true that all men have black hair." This is not the same as the statement, "All men do not have black hair," because this statement implies that *every* man does not have black hair. "Some men do not have black hair" means that *at least* one man does not have black hair and is the negation of "All men have black hair." Consider the following propositions and their negations.

Propositions	*Negations*
Some women have red hair.	All women do not have red hair.
All bananas are yellow.	Some bananas are not yellow.
Some professors are baldheaded.	All professors are not baldheaded.
All students work hard.	Some students do not work hard.