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Spectral Geometry of Manifolds with Boundary and Decomposition of Manifolds

Proceedings of the Workshop on
Spectral Geometry of Manifolds with
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Roskilde University, Roskilde, Denmark
August 6–9, 2003

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Spectral Geometry of Manifolds with Boundary and Decomposition of Manifolds

Preface

On August 6-9, 2003, a workshop was held on *Spectral Geometry of Manifolds with Boundary and Decomposition of Manifolds* at Roskilde University in Denmark.

We envisioned a small meeting with a limited number of lectures and plenty of space and time for interaction between mathematicians having different perspectives on the topic. It is our feeling that the meeting was successful in achieving its primary goals. We think that the spirit of the meeting, and the extremely positive and creative atmosphere among the participants, are reflected in these Proceedings.

The papers in this volume are devoted to presenting developments in the subjects from an interlaced geometric and analytical point of view. They cover a wide variety of topics, from recent advances in index theory and more general theory of spectral invariants on closed manifolds and manifolds with boundary, to applications of those invariants in geometry, topology, and physics.

The papers are grouped in four parts. Part I gives an overview of the subject from various points of view. The survey of Vassilevich places the study of spectral invariants in its physics context in Quantum Field Theory, and Esposito explains recent results on local (differential) problems in Quantum Gravity. Grubb reviews the current results and analysis methods for problems with nonlocal (pseudodifferential projection) boundary conditions.

Part II deals with spectral invariants such as traces, indices and determinants. The paper by Grubb gives a novel deduction of Kontsevich and Vishik's canonical trace on closed manifolds by resolvent methods, obtaining additional features for symbols with a parity or a log-polyhomogeneity property. Lee provides an asymptotic expansion of the zeta-determinant, in terms of the cylinder length, of a Laplacian on a manifold stretched by a cylinder, with various boundary conditions. Park and Wojciechowski show how the quotient between the zeta-determinants of Neumann and Dirichlet problems for a Laplacian identifies with the zeta-determinant of the Dirichlet-to-Neumann operator living on the boundary; this is done precisely and not only in the adiabatic limit.

Part III is concerned with general geometric and topological questions. Boden, Herald and Kirk focus on the spectral flow calculations that constitute an important technical issue in the construction of gauge theoretic Casson-like invariants on 3-manifolds. Leichtnam and Piazza work out a generalization of the classical cut-and-paste result for the signature of closed manifolds to the case of higher signatures of foliated bundles. Lesch settles a variety of topological questions regarding the space of (generally unbounded) self-adjoint Fredholm operators in complex Hilbert space, and shows that there are no other additive and normalized homotopy invariants of paths of such operators than spectral flow. On the basis of their previous work on equivariant Seiberg–Witten Floer cohomologies, Marcolli and Wang deduce a collection of topological invariants for rational homology 3-spheres.

Finally, Part IV deals specifically with problems on manifolds with singularities. Loya shows how Atiyah–Patodi–Singer problems on manifolds with boundary are turned into special cases of problems in the b-calculus (for singular manifolds), for which powerful tools are brought into play in the study of index and eta invariants. Nazaikinskii, Rozenblum, Savin and Sternin attack index problems on manifolds with singularities by methods from K–theory of algebras and cyclic cohomology. The manifolds discussed by Nistor are singular in the sense of having a polyhedral structure at infinity; here spectra and conditions for Fredholmness are determined.

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Part I

Basic Material — Reviews

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Spectral Problems from Quantum Field Theory

Dmitri V. Vassilevich

ABSTRACT. We describe how spectral functions of differential operators appear in the quantum field theory context. We formulate consistency conditions which should be satisfied by the operators and by the boundary conditions. We review some modern developments in quantum field theory and strings and show which new spectral and boundary value problems arise.

1. Introduction

There is no sharp boundary between physics and mathematics. The list of topics which are considered as being parts of these two disciplines varies in space and time. Moreover, there are topics which belong to both. Spectral geometry is just one of the fields where the interaction between physicists and mathematicians has been especially fruitful. On the other hand, stylistic and linguistic differences between traditional physical and mathematical literature are considerable, so that some extremists could even suggest that there is no boundary since there is a gap.

The most immediate aim of this paper is to show how the notions of quantum field theory (QFT) can be translated to the language of spectral theory. Also, I would like to give the reader an idea of which structures can appear in the QFT context, which structures are less likely, and which are forbidden on general grounds

The way spectral functions appear in quantum theory may be illustrated by the following simple example. It is well known that the zero-point energy (the lowest energy level) of the harmonic oscillator with the frequency ω is $E_\omega = \frac{\hbar}{2}\omega$ ($\hbar$ is the Planck constant). Therefore, the ground state energy (lowest energy) of a system of non-interacting harmonic oscillators with eigenfrequencies ω_j is

$$E_0 = \frac{\hbar}{2} \sum_j \omega_j \,. \tag{1.1}$$

QFT is characterised by the presence of an infinite number of degrees of freedom, i.e. it corresponds to an infinite system of harmonic oscillators with eigenfrequencies defined by eigenvalues of a differential operator. The sum (1.1) is typically divergent, but may be regularised by relating it to the zeta function of the operator in question.

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The first spectral function which appeared in QFT was not, however, the zeta function but the heat kernel which was used by Fock [**32**] in 1937 to represent the Green functions. Later, in 1951, this representation was used by Schwinger [**73**] in his famous work on quantum electrodynamics. DeWitt [**23**] made the heat kernel a standard tool to study QFT in curved space-time. In mid 1970s Dowker and Critchley [**26**] and Hawking [**49**] introduced the zeta function regularization thus giving a precise meaning to the idea sketched in the previous paragraph.

During the same period important developments appeared in mathematics as well. The 1949 papers by Minakshisundaram [**59, 60**] had much influence on the theoretical physics research. The Atiyah-Singer Index Theorem [**4**] found many applications in gauge theories. The works of Seeley [**74**] who analysed asymptotics of the spectral functions, and of Gilkey who suggested [**36**] the most effective way for actual calculation of these asymptotics were essential for QFT in curved space-time.

There are many books and review papers which treat the problems of common interest for spectral geometry and quantum field theory[1]. In particular, the monographs by Gilkey [**37**] and Grubb [**44**] contain a very detailed description of relevant mathematics. The book by Kirsten [**52**] deals also with some physical applications as the Casimir energy and the Bose-Einstein condensation. The reviews by Barvinsky and Vilkovisky [**10**] and by Vassilevich [**80**] (see also a shorter version [**79**]), as well as the book by Elizalde [**29**] are oriented to the physicists. In the present paper I introduce some new structures which appeared recently in QFT and may be of interest for the experts in spectral geometry.

This paper is organised as follows. In the next section I briefly introduce the path integral quantisation, the effective action, and the semiclassical expansion. The leading order of this expansion is defined by spectral functions of some differential operator which may be derived from the classical action. I also discuss which properties some general properties of the operator and of the boundary conditions. This section describes objects instead of rigorously defining them. However, one can give precise mathematical sense to most of the constructions presented here. The interested reader can consult the introduction to QFT specially tailored for the mathematicians [**22**]. Sec. 3 discusses main spectral functions appearing in the context of QFT. In particular, the divergences ("infinities") of the effective action are defined by the heat trace asymptotics, and the "finite" part is the zeta-determinant. Quantum anomalies are related to localised zeta functions. In this section we also discuss which properties of the heat trace asymptotics are essential to have a meaningful quantum field theory. Throughout this paper we discuss bosonic fields theories and Laplace type operators. Fermionic theories give rise to operators of Dirac type. An introduction to the theory of Dirac operators with applications to QFT can be found in the monograph [**30**] by Esposito. Sec. 4 contains examples of several new problems which appeared in physics over the recent years. In particular, sec. 4.1 is devoted to string theory. Here we discuss which boundary conditions correspond to open strings, and which to the Dirichlet-branes. We also comment on string dualities and non-commutativity (as it comes from strings). In sec. 4.2 we consider domain walls and the so-called brane-world scenario. These configurations can be described as two smooth manifolds glued together along a

[1] I cannot mention all publications which may seem relevant here or in the text below. I ask the authors whose works are omitted for understanding.

common boundary. Instead of boundary conditions one has matching conditions on the interface surface in this case. Last sections contain remarks on supersymmetric theories and on non-commutative field theories respectively.

I am grateful to Bernhelm Booss-Bavnbek, Gerd Grubb and Krzysztof Wojciechowski for their kind invitation to and warm hospitality at the Spectral Geometry Workshop at Holbaek.

2. Quantum field theory and the path integral

Classical Field Theory consists of (i) a Riemannian manifold M called the space-time, (ii) a Hermitian vector bundle V with sufficiently smooth sections ϕ which are called fields, and (iii) a classical action S defined on ϕ with values in $\mathbb{R}$.

Some comments are in order. Strictly speaking, the space-*time* should be a *pseudo*-Riemannian manifold. Transitions between the Riemannian (Euclidean) and pseudo-Riemannian (Minkowski) signatures of the metric are performed by the so-called Wick rotation which introduces an imaginary time coordinate (relations between spectral theory and the Wick rotation are discussed by Fulling [**33**]). In this manner most (but not all) properties of an Euclidean theory may be translated to the Minkowski context. Besides, Euclidean field theories contain much of important physics and interesting mathematics of their own. In this paper we restrict ourselves to Euclidean theories.

In many cases, classical fields do not form a vector bundle since the fibres may have a more complicated geometry. However, in this work we shall restrict ourselves to the perturbative analysis, i.e. we shall work with small fluctuations about a given background field. Such small fluctuations always form a vector space for any model. We also assume given a Hermitian structure though it is not always uniquely defined. It is natural to assume that the classical action S is bounded from the below[2]. Local minima of the classical action are called classical solutions or classical trajectories of the theory.

Fundamental theories of physics are local, i.e. the action has the form

$$S = \int_M \mathcal{L}_{\text{int}} + \int_{\partial M} \mathcal{L}_{\text{bou}}\,, \tag{2.1}$$

where $\mathcal{L}_{\text{int}}$ and $\mathcal{L}_{\text{bou}}$ depend on fields at a given point and on finite number of their derivatives.

Probably the most popular example of a field theory model is the so called ϕ^4 theory in $n = 4$ dimensions. For simplicity, we suppose that V is a real line bundle. Then the classical action reads:

$$S_{\phi^4} = \int_M \left[(\nabla\phi)^2 + m^2\phi^2 + g\phi^4\right] \tag{2.2}$$

Here m^2 and g are (positive) constants.

To quantise a given classical theory one has to replace the classical fields ϕ by operator valued distributions in a suitably defined Hilbert space. The final aim is to be able to calculate the so-called vacuum expectation values of arbitrary polynomials of the field operators. For these objects there is a "path integral"

[2]Euclidean gravity is a famous exception [**35**].

representation[3]:

$$(2.3)\qquad \langle \phi(x_1)\dots\phi(x_k)\rangle_0 = \frac{1}{N}\int (\mathcal{D}\phi)\,\phi(x_1)\dots\phi(x_k)e^{-\frac{1}{\hbar}S}\,.$$

Here the bracket $\langle\dots\rangle_0$ denotes the vacuum expectation value, $\hbar$ is the Planck constant, $(\mathcal{D}\phi)$ is the integration measure, $1/N$ is a normalisation constant chosen in such a way that $\langle 1\rangle_0 = 1$.

Let me stress that the vacuum expectation values of field polynomials contain practically all information which is needed in QFT. The representation (2.3) has to be derived from basic principles of quantum mechanics, but it can be easily understood on its' own. The right hand side of (2.3) is nothing else than a statistical average with the weight $e^{-\frac{1}{\hbar}S}$. In the formal limit $\hbar\to 0$ the path integral is dominated by the minima of the classical action S thus recovering the classical theory.

The integral (2.3) is infinite-dimensional and, as it stays, is ill-defined. Besides, the measure $(\mathcal{D}\phi)$ may have a very complicated structure. We shall ignore these difficulties in this section and just deal with the path integral as with an ordinary integral. This will be enough to achieve a qualitative understanding of what is going on in quantum field theory.

The vacuum expectation values can be generated by taking repeated functional derivatives of the functional

$$(2.4)\qquad Z(J) = \frac{1}{N}\int (\mathcal{D}\phi)\,e^{-\frac{1}{\hbar}(S+\int_M J\phi)}$$

with respect to the "external source" J.

There is another functional, $W(\bar\phi)$, which is given by the Legendre transform of $-\ln Z(J)$. It is called the effective action and is defined by the equation

$$(2.5)\qquad e^{-\frac{1}{\hbar}W(\bar\phi)} = \frac{1}{N}\int (\mathcal{D}\phi)\,e^{-\frac{1}{\hbar}\left(S(\bar\phi+\phi)+\int_M J\phi\right)}\,,$$

where J is not an independent variable any more. It should be expressed in terms of the *background field* $\bar\phi$ by means of

$$(2.6)\qquad \frac{\delta W(\bar\phi)}{\delta\bar\phi} = -J\,.$$

The effective action W contains the same information as $Z(J)$ but is somewhat easier to analyse. Let us consider a *semiclassical expansion* of W. This means $\hbar\to 0$ asymptotics of the equations (2.5) and (2.6). To this end one has to use the saddle point method to evaluate the integral in (2.5). Let us expand the classical action $S(\bar\phi+\phi)$ about $\bar\phi$,

$$(2.7)\quad S(\bar\phi+\phi) = S(\bar\phi) + \int \frac{\delta S}{\delta\bar\phi(x)}\phi(x) + \frac{1}{2}\int\int \frac{\delta^2 S}{\delta\bar\phi(x)\delta\bar\phi(y)}\phi(x)\phi(y) + O(\phi^3)\,.$$

the zeroth order approximation to the effective action W is just the classical action:

$$(2.8)\qquad W_0(\bar\phi) = S(\bar\phi)\,.$$

[3]A simple and clean derivation of this representation can be found in the excellent text book [**31**] by Faddeev and Slavnov.

Consequently, to this order

$$J_0 = -\frac{\delta S(\bar{\phi})}{\delta \bar{\phi}}\,, \tag{2.9}$$

so that the linear terms in ϕ in the exponential (2.5) are cancelled. The next approximation to the effective action is obtained by keeping the quadratic term in (2.7) and performing the Gaussian integration. Usually there exists a (pseudo)differential operator D such that

$$S_2(\bar{\phi}, \phi) := \int_x \int_y \frac{\delta^2 S(\bar{\phi})}{\delta\bar{\phi}(x)\delta\bar{\phi}(y)} \phi(x)\phi(y) = \int \phi D[\bar{\phi}]\phi\,. \tag{2.10}$$

Therefore,

$$W_1(\bar{\phi}) = \frac{\hbar}{2} \ln \det(D) \tag{2.11}$$

We stress again that we are working with *bosonic* theories only. For fermions the rules of functional integration are considerably different, so that (2.11) is no longer true.

This semiclassical expansion is also called the loop expansion since in the language of Feynman diagrams W_q is described by graphs containing q loops.

One can also define the correlation functions of ϕ in the presence of the background field $\bar{\phi}$. In this case one has to keep the sources unrestricted to be able to vary with respect to them. Then the semiclassical expansion looks a little bit more complicated. In the leading order in $\hbar$ the result reads:

$$\langle \phi(x)\phi(y) \rangle \sim \mathcal{G}(x, y)\,, \tag{2.12}$$

where $\mathcal{G}(x, y)$ is the Green function, $D\mathcal{G} = \mathrm{Id}$.

The operator D should satisfy some natural restrictions:

(1) The operator D should be symmetric. Otherwise, the Gaussian integral would not produce $\det(D)$.
(2) Since the fundamental actions are local (an exception will be discussed in sec. 4.4 below), the operator D is a partial differential operator rather than a pseudodifferential one.
(3) The operator D should have finite number of negative and zero modes. Path integration in the directions corresponding to negative and zero modes cannot be performed by the saddle point method. These directions should be treated separately.

Note, that infinite number of zero modes is a characteristic feature of *gauge theories* (cf. a mathematical introduction [**54**] by Marathe and Martucci). During the quantisation these zero modes are removed by a gauge fixing procedure [**31**], so that the resulting quantum theory satisfies the restriction given above.

If M has a boundary, r.h.s. of (2.10) is supplemented by a boundary term,

$$S_2(\bar{\phi}, \phi) = \int_M \phi D[\bar{\phi}]\phi + \int_{\partial M} \mathcal{L}^2_{\mathrm{bou}}(\bar{\phi}, \phi)\,, \tag{2.13}$$

where the boundary density $\mathcal{L}^2_{\mathrm{bou}}(\bar{\phi}, \phi)$ is quadratic in ϕ. One has to impose some boundary conditions on ϕ. They should be such that (i) the properties (1) and (3) listed above hold, and (ii) the boundary term in (2.13) vanishes, so that the Gaussian integration produces eigenvalues of D at least formally.

Let us now consider the example (2.2). Obviously, the quadratic part of the action reads:

$$S_2(\bar{\phi}, \phi) = \int_M \phi(-\nabla^2 + m^2 + 6g\bar{\phi}^2)\phi - \int_{\partial M} \phi \nabla_n \phi \,, \tag{2.14}$$

where ∇_n is a derivative with respect to inward pointing unit vector on the boundary. Clearly, in this case

$$D = -\nabla^2 + m^2 + 6g\bar{\phi}^2 \tag{2.15}$$

is an elliptic partial differential operator. Natural boundary conditions which ensure vanishing of the boundary term in (2.14) are either Dirichlet

$$\phi|_{\partial M} = 0 \,, \tag{2.16}$$

or Neumann

$$\nabla_n \phi|_{\partial M} = 0 \,, \tag{2.17}$$

ones. Both guarantee strong ellipticity of the boundary value problem and satisfy all consistency conditions listed above. A somewhat less trivial fact is that these conditions are also satisfied by rather complicated mixtures of Dirichlet and Neumann conditions (cf. sec. 4).

3. Spectral functions and QFT

As we have seen in the previous section, first non-trivial quantum correction to classical action is defined by $\det(D)$. This quantity has to be regularised. Let us define the *heat trace* $K(t, D)$ by the equation

$$K(t, D) = \mathrm{Tr}_{L^2}\left(e^{-tD}\right) \,. \tag{3.1}$$

There exist a useful representation for the determinant

$$\ln \det(D) = -\int_0^\infty \frac{dt}{t} K(t, D) \,, \tag{3.2}$$

which is still divergent, but is nevertheless useful to discuss regularizations. One possible way to make sense of (3.2) is to shift the power of t (and introduce a constant $\tilde{\mu}$ of the dimensions of mass to keep proper dimension of the effective action) [**26, 49**]

$$\ln \det(D)_s = -\tilde{\mu}^{2s} \int_0^\infty \frac{dt}{t^{1-s}} K(t, D) \,, \tag{3.3}$$

so that the integral converges for "typical" D for sufficiently large $\mathrm{Re}s$. The zeta function is defined as

$$\zeta(s, D) := \mathrm{Tr}_{L^2}(D^{-s}) \tag{3.4}$$

We assume that D is positive (negative and zero modes should be excluded, cf. previous section). The heat trace is then expressed as

$$K(t, D) = \frac{1}{2\pi i} \int_{\mathrm{Re}s = c} t^{-s} \Gamma(s) \zeta(s, D) \,. \tag{3.5}$$

c should be sufficiently large.

We shall also need a "localised" (or "smeared") version of the heat trace. Let f be a smooth function on M (or an endomorphism of the vector bundle V, depending on the context). Then

$$K(f;t,D) = \mathrm{Tr}_{L^2}\left(fe^{-tD}\right) . \tag{3.6}$$

Obviously, $K(t,D) = K(1;t,D)$. One can also define a localised version of the zeta function, so that the relations between the heat trace and the zeta function will hold also in the local sense.

We assume that the following asymptotic exists for $t \to +0$:

(3.7)

$$K(f;t,D) \simeq t^{-n/2}\left(\sum_{k=0}^{N} t^{k/2} a_k(f,D) + \sum_{j=N+1}^{\infty} t^{j/2}(a'_j(f,D)\ln t + a''_j(f,D))\right),$$

so that we have mixed power and power-logarithm asymptotic expansion. The coefficients a_k are locally computable, i.e. they can be represented as integrals of local invariants constructed from the symbol of D. This expansion indeed exists if D is an elliptic second order operator and if the boundary conditions satisfy some additional requirements formulated by Grubb and Seeley [**47, 45**]. In the particular case when D is of Laplace type and the boundary conditions are local, logarithms are absent ($N = \infty$). If, moreover, M has no boundary, even numbered coefficients vanish.

For any operator of Laplace type there exist a unique connection ∇ and a unique endomorphism E such that

$$D = -(g^{\mu\nu}\nabla_\mu\nabla_\nu + E) \tag{3.8}$$

($g^{\mu\nu}$ is Riemannian metric on M). Let $\partial M = \emptyset$. Let $R_{\mu\nu\rho\sigma}$ be the Riemann curvature tensor, let $R_{\mu\nu} := R^{\sigma}{}_{\mu\nu\sigma}$ be the Ricci tensor, and let $R := R^\mu_\mu$ be the scalar curvature. We define the field strength of $\nabla_\mu = \partial_\mu + \omega_\mu$ by the equation $\Omega_{\mu\nu} := \partial_\mu\omega_\nu - \partial_\nu\omega_\nu + \omega_\mu\omega_\nu - \omega_\nu\omega_\mu$. Then

$$\begin{aligned}
a_0(f,D) &= (4\pi)^{-n/2}\int_M \mathrm{tr}\{f\}. \\
a_2(f,D) &= (4\pi)^{-n/2}6^{-1}\int_M \mathrm{tr}\{f(6E+R)\}. \\
a_4(f,D) &= (4\pi)^{-n/2}360^{-1}\int_M \mathrm{tr}\{f(60\nabla^2 E + 60RE + 180E^2 \\
&\quad +12\nabla^2 R + 5R^2 - 2R_{\mu\nu}R^{\mu\nu} + 2R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} + 30\Omega^{\mu\nu}\Omega_{\mu\nu})\}.
\end{aligned} \tag{3.9}$$

These expressions appeared in both mathematical (MacKean and Singer [**55**]) and physical (DeWitt [**23**]) literature. The coefficient a_6 was first calculated by Gilkey [**36**], a_8 was obtained by Amsterdamski et al[**2**] and by Avramidi [**5**], and a_{10} was calculated by van de Ven [**76**]. In the presence of boundaries the calculations are much more involved. For local mixed boundary conditions a_4 was calculated by Branson and Gilkey [**17**] (with minor corrections by Vassilevich [**77**]), and a_5 was done by Branson et al [**18**].

Equation (3.5) implies that there is a relation between the poles of $\Gamma(s)\zeta(s,D)$ and the asymptotic expansion for the heat trace. In particular, if $a'_n(D) := a'(1,D) = 0$ the zeta function is regular at $s = 0$. This is achieved in the case