CONTEMPORARY MATHEMATICS

460

Toric Topology

International Conference
May 28–June 3, 2006
Osaka City University
Osaka, Japan

Megumi Harada
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Toric Topology

Preface from the editors

1. Overview of the conference

The International Conference on Toric Topology took place at Osaka City University, Japan, from 28 May to 3 June 2006. We took the opportunity to celebrate Akio Hattori's 77th birthday, which is a special anniversary in Japanese culture.

The goals of the conference were to introduce and explore various themes of research in the subject of *toric topology,* and to promote the interaction between people who work on different aspects of torus actions, such as topological, combinatorial, and symplectic- or algebro-geometric.

The four organizers of the conference are also the editors of this proceedings volume.

Over 100 people from about 20 different countries took part in the conference. There were 43 talks total, consisting of 6 plenary talks of 1 hour and 9 invited talks of 45 minutes, in addition to 28 half-hour talks conducted in 2 parallel sessions. We were able to support many young mathematicians, both financially and by providing them with the opportunity to give a talk.

In the remaining part of this preface we try to introduce the reader to two particular aspects of toric topology; in no way do we make the claim that this brief introductory material is complete. We apologize for our many omissions. We note here that the volume also includes an introductory survey on toric topology by Victor Buchstaber and Nigel Ray, in addition to the 25 research and expository papers written by conference speakers and participants. Originally, we had intended to officially separate the contents of this volume into two parts: one a series of research papers, and the other a series of survey articles on current and active research areas. However, it later became evident that such a distinction was at best elusive and likely not profitable; there the matter lays at rest. All manuscripts within this volume went through a strict refereeing process conducted by the editors, and we take this opportunity to thank all the referees for their work.

The Media Center of Osaka City University was an ideal venue for the conference, due to its modern and excellent facilities. We also gratefully acknowledge the support from Japanese 21 Century COE Program *Constitution of wide-angle mathematical basis focused on knots*, the Japan Society for the Promotion of Science, and Osaka City University.

2. Toric varieties and their generalizations

In this section, we first give an overview of the history of the study of manifolds and varieties with torus actions, emphasizing in particular the roots in algebraic and symplectic geometry. We then briefly describe two generalizations of the classical notion of toric varieties which have led to the rich and fruitful developments which together merit our name *toric topology.*

Since the 1970s, the study of torus actions has become increasingly important in various areas of pure mathematics, and has stimulated the formation of many interdisciplinary links between algebraic and symplectic geometry, combinatorial and convex geometry, commutative and homological algebra, differential topology, and homotopy theory. The initial impetus for these developments was provided by the theory of *toric varieties* in algebraic geometry. This theory gives a bijection between, on the one hand, complex algebraic varieties equipped with an action of a complex torus with a dense orbit, and on the other hand, *fans*, which are combinatorial objects. The fan allows one to completely translate various algebraic-geometric notions into combinatorics. A valuable aspect of this theory is that it provides many explicit examples of algebraic varieties, leading to applications in deep subjects such as the resolution of singularities and mirror symmetry. It also points the way toward the deep link between the study of these varieties and the associated combinatorics of their fans.

In symplectic geometry, some of the earlier papers that study a symplectic manifold through a group action and its moment map are Frankel's 1959 paper "Fixed points and torsion on Kähler manifolds" and Smale's 1970 papers "Topology and mechanics". Since the early 1980s there has also been much activity in the field of Hamiltonian group actions, largely due to the groundbreaking convexity theorem of Atiyah and Guillemin-Sternberg, and the exact stationary phase formula of Duistermaat-Heckman. Atiyah–Bott and Berline–Vergne exhibited the latter as a special case of localization in equivariant cohomology, thus placing many of the results in the theory of Hamiltonian group actions into the context of equivariant topology. Delzant, in 1988, showed that if the torus is of half the dimension of the manifold the moment map image determines the manifold up to equivariant symplectomorphism. In symplectic geometry, as in algebraic geometry, one translates various geometric constructions into the language of *convex polytopes* and combinatorics, and a valuable aspect of the field is that it provides a wealth of explicit examples.

Moreover, there is a tight relationship between the toric varieties of algebraic geometry and the symplectic toric manifolds of symplectic geometry: namely, a projective embedding of a smooth toric variety determines a symplectic form and a moment map. The image of the moment map is then a convex polytope that is dual to the fan.

Symplectic toric manifolds are also finding their way into "hard" symplectic topology. We give here a partial sample of recent results: McDuff and Tolman considered a subcircle action as a loop in the group $\mathrm{Ham}(M,\omega)$ of Hamiltonian symplectomorphisms and gave conditions for it to represent a trivial element of $\pi_1(\mathrm{Ham}(M,\omega))$. Tolman and Karshon used torus actions to construct symplectic embeddings of balls and deduced lower bounds on Gromov widths. Biran, Entov, and Polterovich used Floer theory to deduce that certain moment map fibers cannot be displaced by a Hamiltonian isotopy. Karshon, Kessler, and Pinsonnault showed that a compact symplectic four-manifolds admits only finitely many inequivalent toric actions.

We close this brief account of the historical origins of toric topology with some references for further reading. Some accounts of the algebraic-geometric theory are *Toroidal embeddings I* by Kempf, Knudsen, Mumford, and Saint-Donat (1973), *The geometry of toric varieties* by Danilov (1978), *Convex bodies and algebraic geometry* by Oda (1988), and Fulton's *Introduction to toric varieties* (1993). Books on the symplectic theory are Guillemin's *Moment maps and combinatorial invariants of Hamiltonian T^n-spaces* (1994) and Audin's *The topology of torus actions on symplectic manifolds* (1991) and *Torus actions on symplectic manifolds* (2004).

We now take a moment to discuss further the translation between the torus action on these manifolds and the combinatorial objects encoding them. In both the smooth algebraic-geometric and the symplectic situations, the compact torus action is locally isomorphic to the standard action of $(S^1)^n$ on $\mathbb{C}^n$ by rotation of the coordinates. Thus the quotient of the manifold by this action is naturally a manifold with corners, stratified according to the dimension of the stabilizers, and each stratum can be equipped with data that encodes the isotropy torus action along that stratum. Not only does this structure of the quotient provide a powerful means of investigating the action, but some of its subtler combinatorial properties may also be illuminated by a careful study of the equivariant topology of the manifold. Thus, it should be no surprise that since the beginning of the 1990s, the ideas and methodology of toric varieties and Hamiltonian torus actions have started penetrating back into algebraic topology.

The general theory of torus actions has been developed over several decades, and forms an important sub-branch of equivariant topology. As the net of activity has spread wider, and the literature has correspondingly grown, a field of activity has emerged which merits the title *toric topology.* Toric topology is the study of algebraic and symplectic, combinatorial, differential, geometric, and homotopy-theoretic aspects of a particular class of torus actions, whose quotients are highly structured; many of its motivations and guiding principles are provided by the theory of toric varieties and symplectic toric manifolds mentioned above. A characteristic feature is the calculation of topological invariants in terms of combinatorial data associated to these quotients; one of the primary goals of the field is to classify these toric spaces by means of these invariants.

Specifically, during the last two decades, the examples of smooth toric varieties and of symplectic toric manifolds have been generalized into several other classes of manifolds equipped with a torus action. These are mostly of a purely topological nature, in that these more general manifolds are not necessarily algebraic or symplectic. Thus there is more flexibility within these classes of spaces for topological or combinatorial applications. Moreover, they still possess most of the important topological properties of their algebraic or symplectic predecessors. We now describe in more detail two of these generalizations.

First, Davis and Januszkiewicz' influential study of toric varieties from a topological viewpoint, "Convex polytopes, Coxeter orbifolds and torus actions", led to the introduction of the study of *quasitoric manifolds.* These manifolds are determined by two conditions: the T^n-action locally looks like the standard T^n-representation in the complex space $\mathbb{C}^n$, and that the orbit space Q is combinatorially a simple convex polytope. (Both conditions are satisfied for the torus action on a non-singular projective toric variety, but there are examples of quasitoric manifolds which are not toric varieties).

Similarly, Hattori and Masuda's generalization of toric varieties led to a wider class of *torus manifolds.* Apart from the usual conditions on the T^n-action such as smoothness and effectiveness, the fixed point set of a torus manifold is required to be non-empty. Perhaps surprisingly, these more general torus manifolds also admit a combinatorial treatment similar to that of classic toric varieties in terms of fans and polytopes: namely, torus manifolds may be described by *multi-fans* and torus manifolds with equivariant line bundles by *multi-polytopes* as described in Hattori and Masuda, "Theory of multi-fans". The notion of a multi-polytope is a generalization of the twisted polytopes introduced by Karshon-Tolman and is closely related to the notion of a virtual polytope introduced by Khovanskii-Pukhlikov.

As is evident in the brief descriptions above, both quasitoric manifolds and torus manifolds in the sense of Hattori and Masuda still exhibit rich combinatorial structure. Indeed, for either a quasitoric or torus manifold M, the faces of the quotient Q form a simplicial poset S with respect to reverse inclusion, and in the case of quasitoric manifolds, this poset is the poset of faces of a genuine simplicial complex K which is dual to the polytope Q.

3. Combinatorial techniques in toric topology

In addition to the analysis of (multi-)fans and (multi-)polytopes, there are other combinatorial techniques associated to (torus or) toric manifolds. Indeed, an essential component of toric topology is this rich interaction between the topology of the manifolds with various types of combinatorial data associated to them. We describe several themes in this interaction in this section.

3.1. Stanley-Reisner algebras and commutative algebra. Stanley was one of the first to realize the full potential of the subject for combinatorial applications, using it to prove McMullen's conjectured *g-theorem* for face vectors of simplicial polytopes, and the *Upper Bound Conjecture* for triangulated spheres. Commutative algebra and homological algebra have been intertwined with combinatorial geometry ever since Stanley's seminal work.

Many of Stanley's ideas extend to the more general topological context introduced in the previous section. For instance, the *face ring* or *Stanley–Reisner algebra* $\mathbb{Z}[K]$ of K is a crucial ingredient in the computation of the integral cohomology ring of a quasitoric manifold M. Davis and Januszkiewicz perform this calculation by associating an auxiliary T^m-space $\mathcal{Z}_K$ to any complex K on m vertices, and considering its homotopy quotient (or Borel construction) $DJ(K)$. Their definition of $\mathcal{Z}_K$ is inspired by Vinberg's universal space for reflection groups (and so is analogous to that of the *Coxeter complex*). Davis and Januszkiewicz show that the cohomology ring $H^*(DJ(K))$ (or the *equivariant cohomology* $H^*_T(M)$) is isomorphic to $\mathbb{Z}[K]$ for any K. They also deduce that the ordinary cohomology $H^*(M)$ is obtained from $\mathbb{Z}[K]$ by factoring out certain linear forms, exactly as in the situation with toric varieties.

Motivated by Stanley's fundamental contributions, much subsequent work has since confirmed that the face ring $\mathbb{Z}[K]$ encodes many subtle combinatorial properties of K. The study of these algebras has now gained momentum and topological significance independent of its historical origins, and has added a geometrical flavour to the well-developed study of *Cohen–Macaulay rings*, in particular through the notion of a *Cohen–Macaulay complex* K, whose face ring $\mathbb{Z}[K]$ is Cohen–Macaulay, Similarly, the homological viewpoint leads to the study of the bigraded vector spaces $\mathrm{Tor}_{k[v_1,\ldots,v_m]}(k[K],k)$, whose dimensions of bigraded components are known as the algebraic Betti numbers of $k[K]$, for any field k. These numbers are quite subtle invariants: they depend on a combinatorics of K rather than on the topology of its realization $|K|$, and fully determine the "ordinary" topological Betti numbers of $|K|$. Hochster's theorem gives an expression for the algebraic Betti numbers in terms of the homology of full subcomplexes of K.

3.2. GKM manifolds and GKM graphs. In their influential 1998 paper "Equivariant cohomology, Koszul duality, and the localization theorem", Goresky, Kottwitz, and MacPherson computed the equivariant cohomology of certain complex torus actions on algebraic varieties through the combinatorics of a labeled graph, called the *GKM graph*, that is associated to the variety. Their assumptions imply that the action has only finitely many fixed points and finitely many complex one-dimensional orbits. The vertices of the graph then correspond to the fixed points; the edges correspond to the one-dimensional orbits, and they are labeled by corresponding weights of the torus.

Subsequent work shows that such a labeled graph can be constructed under fairly general assumptions, for instance, when a torus T^k acts effectively on a compact $2n$-dimensional manifold M $(k \leq n)$ such that the fixed point set is finite, M possesses an invariant almost complex structure, and the weights of the tangential T^k-representations at the fixed points are pairwise linearly independent. These graphs in these more general situations are also known as GKM graphs (or *moment graphs*). Many geometric properties of Hamiltonian torus actions translate to combinatorial properties of these graphs. The study of these graphs has become of independent combinatorial interest, pursued by Guillemin, Zara, and others.

These graphs also connect to earlier work, of Ahara-Hattori and of Audin, in which they use the combinatorics of Hamiltonian circle actions on compact symplectic four-manifolds in order to characterize these manifolds up to equivariant diffeomorphism. Karshon extended this to a classification up to equivariant symplectomorphisms, still by labeled graphs. All these graphs are part of Tolman's notion of an *X-ray* of a Hamiltonian torus action.

3.3. The moment-angle complex. The study of *moment-angle complexes* is one of the key ingredients in most modern applications of toric topology. It is also closely related to the study of quasitoric manifolds. In their influential work mentioned in Section 2, Davis and Januszkiewicz assigned an auxiliary T^m-space $\mathcal{Z}_K$, the moment-angle complex, to an arbitrary simplicial complex K on m vertices. It has become evident that these spaces $\mathcal{Z}_K$ are of great independent interest in toric topology. For instance, they arise in homotopy theory as *homotopy colimits*, in symplectic geometry as level surfaces for the moment maps for linear torus actions, and in the theory of arrangements as *complements of coordinate subspace arrangements*. The construction of moment-angle complexes gives rise to a functor from the category of simplicial complexes and simplicial maps to the category of spaces with torus actions and equivariant maps. This functor provides an effective way to study invariants of triangulations by the methods of equivariant topology. If K is a triangulation of an $(n-1)$-dimensional sphere, then $\mathcal{Z}_K$ is an $(m+n)$-dimensional manifold. Moreover, if K is a dual triangulation of the boundary of a simple P, then for an arbitrary quasitoric manifold M^{2n} with orbit space P, there is a principal T^{m-n}-bundle $\mathcal{Z}_K \to M^{2n}$.

Buchstaber and Panov computed the cohomology ring of a moment-angle complex in terms of the combinatorics of the simplicial complex K. The cohomology algebra $H^*(\mathcal{Z}_K)$ is shown to be isomorphic to the Tor-*algebra* $\mathrm{Tor}_{\mathbb{Z}[v_1,\dots,v_m]}(\mathbb{Z}[K],\mathbb{Z})$, where $\mathbb{Z}[K]$ is the face ring of K. Through this isomorphism the canonical "algebraic" bigrading in Tor acquires a geometrical realization; it corresponds to a "topological" bigrading of cells in a certain cellular decomposition of $\mathcal{Z}_K$. A further analysis leads to an effective description of the Tor-algebra in terms of the Koszul complex, which allows us to use computer packages like `Macaulay2` or `BISTELLAR` for calculations in combinatorial homological algebra and combinatorial geometry.

4. Concluding remarks

We hope that this quick overview of both the background history and the most recent developments and problems of toric topology has whetted the reader's appetite for more. Luckily, the manuscripts of this volume contain a remarkable breadth of topics further developing the themes we have only been able to sketch above. Indeed, the wide variety of topics covered within these pages only serves to illustrate the fact that toric topology is a rapidly growing subject with many fruitful interactions between a great number of fields. We hope that this volume may serve as an enticing invitation to the subject.

Megumi Harada, Yael Karshon, Mikiya Masuda, and Taras Panov

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