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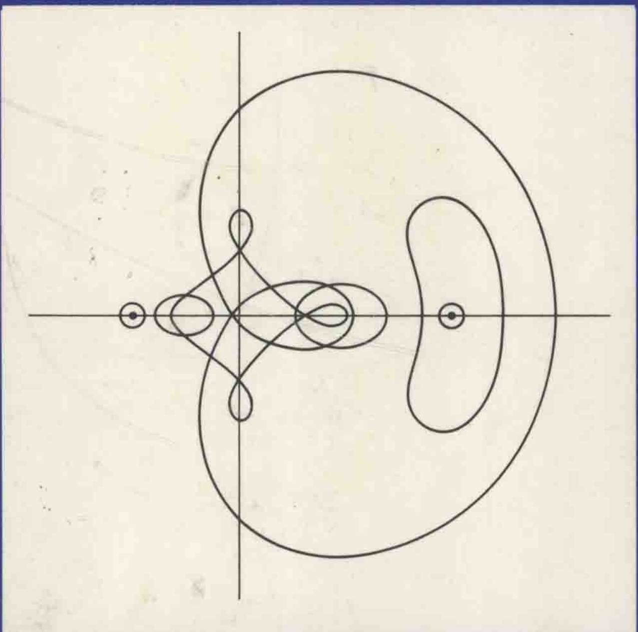
COMPUTATION AND THEORY IN ORDINARY DIFFERENTIAL EQUATIONS



COMPUTATION AND THEORY IN ORDINARY DIFFERENTIAL EQUATIONS

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COMPUTATION AND THEORY IN ORDINARY DIFFERENTIAL EQUATIONS

A SERIES OF BOOKS IN MATHEMATICS

Editors: R. A. ROSENBAUM
G. PHILIP JOHNSON

to our wives, Linda and Judy

PREFACE

Some of this material was first presented by one of the authors (R.E.M.) as a one-semester graduate level topics course in numerical analysis during the spring of 1966 in the Computer Sciences Department at the Madison campus of the University of Wisconsin; the present text represents an expanded version of the lecture notes for that course. Our intent is to present briefly the theoretical background in differential equations and the modern computational methods of solving such equations in such a way as to emphasize the interaction between the two topics. Thus, this book is not a traditional text in either the theory of differential equations or numerical analysis alone; no theorems are stated and proved in detail. Our primary goal is to indicate some ways in which theoretical concepts and computation can be combined to attack a problem efficiently.

With this goal in mind, we have divided the material into three parts. The first two parts comprise a survey of abstract theory and computational methods that use theoretical insights. In the third part we examine transformations that can be used as analytical and computational tools to solve many kinds of problems; it is one of our main contentions that pre-analysis and transformations which utilize partial knowledge about the solutions are of great value. We are primarily interested in initial value problems, especially in Part 3, although we do examine boundary value problems as well; eigenvalue problems are not included at all.

Although much of this text could be understood by a person without knowledge of either differential equations or numerical analysis, it is intended for an audience aware of the nature of both of these subjects and interested in studying their interactions in order to achieve more accurate and efficient computation. We hope that this material will help raise interest in the general application of analytical techniques to computations; we would especially like to see the growth of machine-automated analytical methods, which have only recently become feasible.

Thanks are due—and are herewith given to—our colleague, Ben Noble, whose many suggestions are invaluable.

Madison, Wisconsin
November, 1969

James W. Daniel
Ramon E. Moore

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NOTATIONAL CONVENTIONS

A bibliography of books and articles cited in the text begins on page 165: throughout the text these are referred to by the numbers given them in the bibliography. References to parts of this text itself take the form "Section 2.3," which refers to the third titled section in Chapter 2, or the form "Eq. 2–3." which refers to the third numbered equation in Chapter 2.

Vectors will be denoted both as $(y_1, \ldots, y_n)$ and $\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$. If $\mathbf{y}$ is a vector and $\mathbf{f}$ a vector-valued function of $\mathbf{y}$, $\mathbf{f}(\mathbf{y})$ denotes $(f_1(y_1, \ldots, y_n), \ldots, f_n(y_1, \ldots, y_n))$. The matrix $\partial \mathbf{f}/\partial \mathbf{y}$ has as its i, j element the scalar $\partial f_i/\partial y_j$.

MATHEMATICAL SYMBOL	MEANING
E^n	n-dimensional real Euclidean space
$\lVert \cdot \rVert$	An arbitrary vector norm and the generated matrix norm
$\lVert \cdot \rVert_\infty$	The maximum norm For vectors: $\lVert \mathbf{x} \rVert_\infty = \max\limits_{1 \leq i \leq n} \lvert x_i \rvert$ For matrices: $\lVert \mathbf{A} \rVert_\infty = \max\limits_{1 \leq i \leq n} \sum\limits_{j=1}^{n} \lvert A_{ij} \rvert$
$\in$	Is a member of
$\subset$	Is a subset of
$'$ (prime)	$\dfrac{d}{dt}$
$u(h) = O(h)$	There is a constant K such that $\lvert u(h) \rvert \leq K \lvert h \rvert$ as $h \longrightarrow 0$
$u(h) = o(h)$	$\lim\limits_{h \longrightarrow 0} \left(\dfrac{u(h)}{h} \right) = 0$

CONTENTS

PART 1

THEORETICAL BACKGROUND

Many mathematical problems that arise in practice do not fit into neat classes that can be treated by a well-developed theory. A reasonable man, however, should make use of any applicable theory whenever possible in order to acquire some understanding of the behavior of the solution to his problem. In these first three chapters we survey briefly some of the theoretical results that we feel can most often contribute to reducing the overall effort of computing a solution. We by no means claim to review everything that might be useful.

Chapter 1 discusses geometric concepts related to differential equations, and therefore dwells briefly on such topics as curves, surfaces, trajectories, integral surfaces, and vector fields. Chapter 2 examines several aspects of initial value problems, including the existence, uniqueness, and representation of solutions; dependence of the solutions on the initial data and on the differential equation (perturbation theory); and the asymptotic behavior of solutions. The chapter concludes with a few remarks on periodic solutions.

Existence, uniqueness, and representation of solutions to boundary value problems are treated in Chapter 3, along with reformulations of the problem as an integral equation or variational problem; we also discuss a relevant monotonicity property.

1

BASIC CONCEPTS; GEOMETRY

1.1 INTRODUCTION

We are interested in ordinary differential equations, that is, in equations involving the derivative of an unknown function with respect to a single variable t, often understood to represent time. In particular, we often will consider equations of the form

$$\mathbf{y}' = \mathbf{f}(\mathbf{y}).$$

If $y = y(t)$ is a real-valued function of the real variable t, the differentia equation tells y whether it should increase ($y' = f(y) \geq 0$) or decrease ($y' = f(y) \leq 0$), and how rapidly it should do either (rapidly if $|y'| = |f(y)|$ is large) at any value of y.

More generally, if $\mathbf{y}$ and $\mathbf{f}$ are vectors in n-dimensional Euclidean space E^n, so that $\mathbf{y} = (y_1, \ldots, y_n)$ and $\mathbf{f}(\mathbf{y}) = (f_1(y_1, \ldots, y_n), \ldots, f_n(y_1, \ldots, y_n))$, the differential equation still describes how at any point $\mathbf{y}$ the vector $\mathbf{y}$ must modify itself in order to qualify as a solution. Just as in one dimension a positive or negative value for $f(y)$ means "increase" or "decrease," the vector $\mathbf{f}(\mathbf{y})$ indicates the direction in which $\mathbf{y}$ must move, and the magnitude of $\mathbf{f}(\mathbf{y})$ indicates speed. Such an intuitive geometrical view of differential equations is often useful; we pursue these notions further.

1.2 VECTOR FIELDS; DIFFERENTIABLE CURVES AND SURFACES

Let $\mathbf{Y}$ be a set of vectors $\mathbf{y} = (y_1, \ldots, y_n)$ in E^n. A *vector field* on $\mathbf{Y}$ is a mapping $\mathbf{f} = (f_1, \ldots, f_n)$ assigning a vector $\mathbf{f}(\mathbf{y}) \in E^n$ to each $\mathbf{y} \in \mathbf{Y}$. The vector field is said to be continuous if the mapping **f** is continuous; this is equivalent to continuity for each of the real-valued functions $f_i(\mathbf{y})$, $i = 1, 2, \ldots, n$. Perhaps the simplest way to describe a vector field graphically is

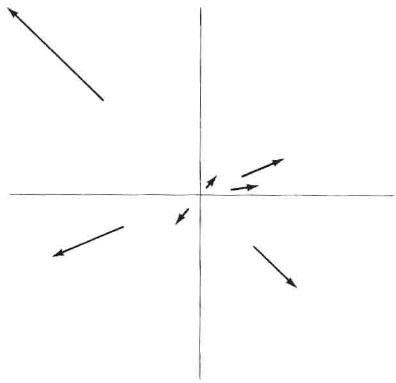

FIGURE 1–1. $\mathbf{f}(y_1, y_2) = (y_1, y_2)$

to draw the vector $\mathbf{f}(\mathbf{y})$ as emanating from the point **y**. Thus, Figures 1–1 through 1–4 depict vector fields on E^2, although not all possible vectors are drawn.

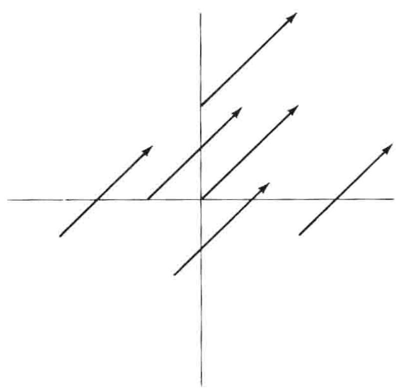

FIGURE 1–2. $\mathbf{f}(y_1, y_2) = (1,1)$

The first three vector fields are continuous; the fourth is not, since $f_1(y_1, y_2)$ is discontinuous along the line $y_1 = 1$. The third vector field looks somewhat discontinuous along the unit circle because of the directions of the field near there; the limit from inside or outside the circle is the same, however, saving the day. The limit is, in fact, zero (i.e., (0, 0)), and the field is said to *vanish* at such a point. The field in Figure 1–1 vanishes only at $\mathbf{y} = (0, 0)$; that in Figure 1–2 never vanishes; that in Figure 1–3 vanishes at $y = (0, 0)$ as well as along $y_1^2 + y_2^2 = 1$; and that in Figure 1–4 vanishes along $y_1 = 0$.