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FINANCIAL MODELS WITH LÉVY PROCESSES AND VOLATILITY CLUSTERING

SVETLOZAR T. RACHEV • YOUNG SHIN KIM
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STR
To my grandchildren Iliana, Zoya, and Svetlozar

YSK
To my wife Myung-Ja and my son Minseob

MLB
To my wife Giorgia

FJF
To my wife Donna and my children Francesco, Patricia, and Karly

Preface

Carl Frederick Gauss, born in 1777, is one of the foremost mathematicians the world has known. Labeled the "prince of mathematicians" and viewed by some as on par with Sir Isaac Newton, the various works of Gauss have influenced a wide range of fields in mathematics and science. Although very few in the finance profession are familiar with his great contributions and body of work—which are published by the by the Royal Society of Göttingen in seven quatro volumes—most are familiar with his important work in probability theory that bears his name: the Gaussian distribution. The more popular name for this distribution is the normal distribution and was also referred to as the "bell curve" in 1733 by Abraham de Moivre, who first discovered this distribution based on his empirical work. Every finance professional who has taken a probability and statistics course has had a heavy dose of the Gaussian distribution and probably can still recite some properties of this distribution.

The normal distribution has found many applications in the natural sciences and social sciences. However, there are those who have long warned about the misuse of the normal distribution, particularly in the social sciences. In a 1981 article in *Humanity and Society*, Ted Goertzel and Joseph Fashing ("The Myth of the Normal Curve: A Theoretical Critique and Examination of its Role in Teaching and Research") argue that

> *The myth of the bell curve has occupied a central place in the theory of inequality . . . Apologists for inequality in all spheres of social life have used the theory of the bell curve, explicitly and implicitly, in developing moral rationalizations to justify the status quo. While the misuse of the bell curve has perhaps been most frequent in the field of education, it is also common in other areas of social science and social welfare.*

A good example is in the best-selling book by Richard Herrnstein and Charles Murray, *The Bell Curve*, published in 1994 with the subtitle *Intelligence and Class Structure in American Life*. The authors argue based on their empirical evidence that in trying to predict an individual's income or job performance, intelligence is a better predictor than the educational level or socioeconomic status of that individual's parents. Even the likelihood to commit a crime or to exhibit other antisocial behavior is better predicted by intelligence, as measured by IQ, than other potential explanatory factors. The policy implications drawn from the book are so profound that they set off a flood of books both attacking and supporting the findings of Herrnstein and Murray.

In finance, where the normal distribution was the underlying assumption in describing asset returns in major financial theories such as the capital asset pricing theory and option pricing theory, the attack came in the early 1960s from Benoit Mandelbrot, a mathematician at IBM's Thomas J. Watson Research Center. Although primarily known for his work in fractal geometry, the finance profession was introduced to his study of returns on commodity prices and interest rate movements that strongly rejected the assumption that asset returns are normally distributed. The mainstream financial models at the time relied on the work of Louis Bachelier, a French mathematician who at the beginning of the 20th century was the first to formulate random walk models for stock prices. Bachelier's work assumed that relative price changes followed a normal distribution. Mandelbrot, however, was not the first to attack the use of the normal distribution in finance. As he notes, Wesley Clair Mitchell, an American economist who taught at Columbia University and founded the National Bureau of Economic Research, was the first to do so in 1914. The bottom line is that the findings of Mandelbrot that empirical distributions do not follow a normal distribution led a leading financial economist, Paul Cootner of MIT, to warn the academic community that Mandelbrot's finding may mean that "past econometric work is meaningless."

The overwhelming empirical evidence of asset returns in real-world financial markets is that they are not normally distributed. In commenting on the normal distribution in the context of its use in the social sciences, "Earnest Ernest" wrote the following in the November 10, 1974, in the *Philadelphia Inquirer*:

> *Surely the hallowed bell-shaped curve has cracked from top to bottom. Perhaps, like the Liberty Bell, it should be enshrined somewhere as a memorial to more heroic days.*

Finance professionals should heed the same advice when using the normal distribution in asset pricing, portfolio management, and risk management.

In Mandelbrot's attack on the normal distribution, he suggested that asset returns are more appropriately described by a non-normal stable distribution referred to as a stable Paretian distribution or alpha-stable distribution (α-stable distribution), so-named because the tails of this distribution have Pareto power-type decay. The reason for describing this distribution as "non-normal stable" is because the normal distribution is a special case of the stable distribution. Because of the work by Paul Lévy, a French mathematician who introduced and characterized the non-normal stable distribution, this distribution is also referred to as the Lévy stable distribution and the Pareto-Lévy stable distribution. (There is another important contribution to probability theory by Lévy that we apply to financial modeling in this book. More specifically, we will apply the Lévy processes, a continuous-stochastic process.)

There are two other facts about asset return distributions that have been supported by empirical evidence. First, distributions have been observed to be skewed or nonsymmetric. That is, unlike in the case of the normal distribution where there is a mirror imaging of the two sides of the probability distribution, typically in a skewed distribution, one tail of the distribution is much longer (i.e., has greater probability of extreme values occurring) than the other tail of the probability distribution. Probability distributions with this attribute are referred to as having fat tails or heavy tails. The second finding is the tendency of large changes in asset prices (either positive or negative) to be followed by large changes, and small changes to be followed by small changes. This attribute of asset return distributions is referred to as volatility clustering.

In this book, we consider these well-established facts about asset return distributions in providing a framework for modeling the behavior of stock returns. In particular, we provide applications to the financial modeling used in asset pricing, option pricing, and portfolio/risk management. In addition to explaining how one can employ non-normal distributions, we also provide coverage of several topics that are of special interest to finance professionals.

We begin by explaining the need for better financial modeling, followed by the basics of probability distributions—the different types of probability distributions (discrete and continuous), specific types of probability distributions, parameters of a probability distribution, and joint probability distributions. The definition of the stable Pareto distribution (we adopted the term α-stable distribution in this book) that Mandelbrot suggested is described. Although this distribution has certain desirable properties and is superior to the normal distribution, it is not suitable in certain financial modeling applications such as the modeling of option prices because the mean, variance, and exponential moments of the return distribution have to exist. For this reason, we introduce distributions that we believe are better suited for financial modeling, distributions obtained by tempering the tail properties of the α-stable distribution: the smoothly truncated stable distribution and various types of tempered stable distributions. Because of their important role in the applications in this book, we review continuous-time stochastic processes with emphasis on Lévy processes.

There are chapters covering the so-called exponential Lévy model, and we study this continuous-time option pricing model and analyze the change of measure problem. Prices of plain vanilla options are calculated with both analytical and Monte Carlo methods.

After examples dealing with the simulation of non-normal random numbers, we study two multivariate settings that are suitable to explain joint extreme events. In the first approach, we describe a multivariate random variable for joint extreme events, and in the second we model the joint behavior of log-returns of stocks by considering a feasible dependence structure together with marginals able to explain volatility clutering.

Then we get into the core of the book where we deal with examples of discrete-time option pricing models. Starting from the classic normal model with volatility clustering, we progress to the more recent models that jointly consider volatility clustering and heavy tails. We conclude with a non-normal GARCH model to price American options.

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