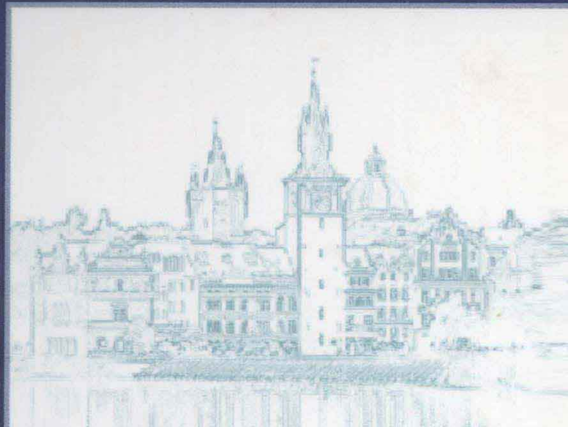
WAVELET THEORY

An Elementary Approach with Applications

DAVID K. RUCH and PATRICK J. VAN FLEET



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An Elementary Approach With Applications

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WAVELET THEORY

To Pete and Laurel
for a lifetime of encouragement
(DKR)

To Verena, Sam, Matt, and Rachel
for your unfailing support
(PVF)

Preface

This book presents some of the most current ideas in mathematics. Most of the theory was developed in the past twenty years, and even more recently, wavelets have found an important niche in a variety of applications. The filter pair we present in Chapter 8 is used by JPEG2000 [59] and the Federal Bureau of Investigation [8] to perform image and fingerprint compression, respectively. Wavelets are also used in many other areas of image processing as well as in applications such as signal denoising, detection of the onset of epileptic seizures [2], modeling of distant galaxies [3], and seismic data analysis [34, 35].

The development and advancement of the theory of wavelets came through the efforts of mathematicians with a variety of backgrounds and specialties, and of engineers and scientists with an eye for better solutions and models in their applications. For this reason, our goal was to write a book that provides an introduction to the essential ideas of wavelet theory at a level accessible to undergraduates and at the same time to provide a detailed look at how wavelets are used in "real-world" applications. Too often, books are heavy on theory and pay little attention to the details of application. For example, the discrete wavelet transform is but one piece of an image compression algorithm, and to understand this application, some attention must be given to quantization and coding methods. Alternatively, books might provide a detailed description of an application that leaves the reader curious about the theoretical foundations of some of the mathematical concepts used in the model. With this book, we have attempted to balance these two competing yet related tenets, and it is ultimately up to the reader to determine if we have succeeded in this endeavor.

To the Student

If you are reading this book, then you are probably either taking a course on wavelets or are working on your own to understand wavelets. Very often students are naturally curious about a topic and wish to understand quickly their use in applications. Wavelets provide this opportunity — the discrete Haar wavelet transformation is easy to understand and use in applications such as image compression. Unfortunately, the discrete Haar wavelet transformation is *not* the best transformation to use in many applications. But it does provide us with a concrete example to which we can refer as we learn about more sophisticated wavelets and their uses in applications. For this reason, you should study carefully the ideas in Chapters 3 and 4. They provide a framework for all that follows. It is also imperative that you develop a good working knowledge of the Fourier series and transformations introduced in Chapter 2. These ideas are very important in many areas of mathematics and are the basic tools we use to construct the wavelet filters used in many applications.

If you are a mathematics major, you will learn to write proofs. This is quite a change from lower-level mathematics courses where computation was the main objective. Proof-writing is sometimes a formidable task and the best way to learn is to practice. An indirect benefit of a course based on this book is the opportunity to hone your proof-writing skills. The proofs of most of the ideas in this book are straightforward and constructive. You will learn about proof by induction and contraposition. We have provided numerous problems that ask you to complete the details of a portion of a proof or mimic the ideas of one case in a proof to complete another. We strongly encourage you to tackle as many of these problems as possible. This course should provide a good transition from the proofs you see in a sophomore linear algebra course to the more technical proofs you might see in a real analysis course.

Of course, the book also contains many computational problems as well as problems that require the use of a computer algebra system (CAS). It is important that you learn how to use a CAS — both to solve problems and to investigate new concepts. It is amazing what you can learn by taking examples from the book and using a CAS to understand them or even change them somewhat to see the effects. We strongly encourage you to install the software packages described below and to visit the course Web site and work through the many labs and projects that we have provided.

To the Instructor

In this book we focus on bridging the gap often left between discrete wavelet transformations and the traditional multiresolution analysis-based development of wavelet theory. We provide the instructor with an opportunity to balance and integrate these ideas, but one should be wary of getting bogged down in the finer details of either topic. For example, the material on Fourier series and transforms is a place where instructors should use caution. These topics can be explored for entire semesters, and deservedly so, but in this course they need to be treated as tools rather than the thrust of the course.

The heart of wavelet theory is covered in Chapters 3, 5, and 6 in a comprehensive approach. Extensive details and examples are given or outlined via problems, so students should be able to gain a full understanding of the theory without hand-waving at difficult material. Having said that, some proofs are omitted to keep a nice flow to the book. This is not an introductory analysis book, nor is the level of rigor up to that of a graduate text. For example, the technical proofs of the completeness and separation properties of multiresolution analyses are left to future courses. The order of infinite series and integration are occasionally swapped with comment but not rigorous justification. We choose not to develop fully the theory of Riesz bases and how they lead to true dual multiresolutions of $L^2(\mathbb{R})$, for this would leave too little time for the very real applications of biorthogonal filters. We hope students will whet their appetites for future courses from the taste of theory they are given here!

We feel that the discrete wavelet transform material is essential to the spirit of the book, and based on our experience, students will find the applications quite gratifying. It may be tempting to expand on our introduction to these ideas after the Haar spaces are built, but we hope sufficient time is left for the development of multiresolution analyses and the Daubechies wavelets, which can take considerable time.

We also hope that instructors will take the time for thorough treatment of the connections between standard wavelet theory and discrete wavelet transforms. Our experience, both personally and with teaching other faculty at workshops, is that these connections are very rewarding but are not obvious to most beginners in the field. Some interesing problems crop up as we move between $L^2(\mathbb{R})$ and finite-dimensional approximations.

Text Topics

In Chapter 1, we provide a quick introduction to the complex plane and $L^2(\mathbb{R})$, with no prior experience assumed, emphasizing only the properties that will be needed for wavelet development. We believe that the fundamentals of wavelets can be studied in depth without getting into the intricacies of measure theory or the Lesbesgue integral, so we discuss briefly the measure of a set and convergence in norm versus pointwise convergence, but we do not dwell heavily on these ideas.

In Chapter 2 we present Fourier series and the Fourier transform in a limited and focused fashion. These ideas and their properties are developed only as tools to the extent that we need them for wavelet analysis. Our goal here is to prepare quickly for the study of wavelets in the transform domain. For example, the transform rules on translation and dilation are given, since these are critical for manipulating scaling function symbols in the transform domain. B-splines are introduced in this chapter as an important family of functions that will be used throughout the book, especially in Chapter 8.

In Chapter 3 we begin our study of wavelets in earnest with a comprehensive examination of Haar spaces. All the major ideas of multiresolution analysis are here, cast in the accessible Haar setting. The properties and standard notations of approximation spaces V_j and detail spaces W_j are developed in detail with numerous examples.

Students may be ready for some applications after the long Haar space analysis, and we present some classics in Chapter 4. The ideas behind filters and the discrete Haar wavelet transform are introduced first. The basics of processing signals and images are developed in Sections 4.1 and 4.2, with sufficient detail so that students can carry out the calculations and fully understand what software is doing while processing large images. The attractive and very accessible topics of image compression and edge detection are introduced as applications in Section 4.3.

In Chapter 5 we generalize the Haar space concepts to a general multiresolution analysis, beginning with the main properties in the time domain. Section 5.2 begins the development of critical multiresolution properties in the transform domain. In Section 5.3 we present some concrete examples of functions satisfying multiresolution properties. In addition to Haar, the Shannon wavelet and B-splines are discussed, each of which has some desirable properties but is missing others. This also provides some motivation for the formidable challenge of developing Daubechies wavelets. We return to B-splines in Chapter 8.

Chapter 6 centers on the Daubechies construction of continuous, compactly supported scaling functions. After a detailed development of the ideas, a clear algorithm is given for the construction. The next two sections are devoted to the cascade algorithm, which we delay presenting until after the Daubechies construction, with the motivation of plotting these amazing scaling functions with only a dilation equation to guide us. The cascade algorithm is introduced in the time domain, where examples make it intuitively clear, and is then discussed in the transform domain. Finally, we study the practical issue of coding the algorithm with discrete vectors.

After the rather heavy theory of Chapters 5 and 6, an investigation of the discrete Daubechies wavelet transform and applications in Chapter 7 provides a nice change of pace. An important concept in this chapter is that of handling the difficulties encountered when the decomposition and reconstruction formula are truncated, which are investigated in Section 7.2. Our efforts are rewarded with applications to image compression, noise reduction and image segmentation in Section 7.3.

In Chapter 8 we introduce scaling functions and wavelets in the biorthogonal setting. This is a generalization of an orthogonal multiresolution analysis with a single scaling function to a dual multiresolution analysis with a pair of biorthogonal scaling functions. We begin by introducing several new ideas via an example from B-splines, with an eye toward creating symmetric filters to be used in later applications. The main structural framework for dual multiresolution analyses and biorthogonal wavelets is developed in Section 8.2. We then move to constructing a family of biorthogonal filters based on B-splines using the methods due to Ingrid Daubechies in Section 8.3. The Cohen–Daubechies–Feauveau CDF97 filter pair is used in the JPEG2000 and FBI fingerprint compression standards, so it is natural to include them in the book. The method of building biorthogonal spline filters can be adjusted fairly easily to create the CDF97 filter pair, and this construction is part of Section 8.3. The pyramid algorithm can be generalized for the biorthogonal setting and is presented in Section 8.4. The discrete biorthogonal wavelet transform is discussed in Section 8.5. An advantage of biorthogonal filter pairs is that they can be made symmetric, and this desirable property affords a method, also presented in Section 8.5, of dealing with edge conditions in signals or digital images. A fundamental theoretical underpinning of dual multiresolution analyses is the concept of a Riesz basis, which is a generalization of orthogonal bases. The very formidable specifics of Riesz bases have been suppressed throughout most of this chapter in an effort to provide a balance between theory and applications. As a final and optional topic in this chapter, a brief examination of Riesz bases is provided in Section 8.6.

Wavelet packets, the topic of Chapter 9, provide an alternative wavelet decomposition method but are more computationally complex since the decomposition includes splitting the detail vectors as well as the approximations. We introduce wavelet packet functions in Section 9.1 and wavelet packet spaces in Section 9.2. The discrete wavelet packet transform is presented in Section 9.3 along with the best basis algorithm. The wavelet packet decomposition allows for redundant representations of the input vector or matrix, and the best basis algorithm chooses the "best" representation. This is a desirable feature of the transformation as this algorithm can be made application-dependent. The FBI fingerprint compression standard uses the CDF97 biorthogonal filter pair in conjunction with a wavelet packet transformation, and we outline this standard in Section 9.4.

Prerequisites

The minimal requirements for students taking this course are two semesters of calculus and a course in sophomore linear algebra. We use the ideas of bases, linear independence, and projection throughout the book so students need to be comfortable with these ideas before proceeding. The linear algebra prerequisite also provides the necessary background on matrix manipulations that appear primarily in sections dealing with discrete transformations. Students with additional background in Fourier series or proof-oriented courses will be able to move through the material at a much faster pace than will students with the minimum requirements. Most proofs in the book are of a direct and constructive nature, and some utilize the concept of mathematical induction. The level of sophistication assumed increases steadily, consistent with how students should be growing in the course. We feel that reading and writing proofs should be a theme throughout the undergraduate curriculum, and we suggest that the level of rigor in the book is accessible by advanced juniors or senior mathematics students. The constant connection to concrete applications that appears throughout the book should give students a good understanding of why the theory is important and how it is implemented. Some algorithms are given and experience with CAS software is very helpful in the course, but significant programming experience is not required.

Possible Courses for this Book

The book can serve as a stand-alone introduction to wavelet theory and applications for students with no previous exposure to wavelets. If a brisk pace is kept in line with the prerequisites discussed above, the course could include the first six chapters plus the discrete Daubechies transform and a sample of its applications. While considerable time can be spent on applied projects, we strongly recommend that any course syllabus include Chapter 6, on Daubechies wavelets. The construction of these wavelets is a remarkable mathematical achievement accomplished during our lifetime (if not those of our students) and should be covered if at all possible.

Some instructors may prefer to first cover Chapters 3 and 4 on Haar spaces before introducing the Fourier material of Chapter 2. This approach will work well since aside from a small discussion of the Fourier series associated with the Haar filter, no ideas from Fourier analysis are used in Chapters 3 and 4.

A very different course can be taught if students have already completed a course using Van Fleet's book *Discrete Wavelet Transformations: An Elementary Approach with Applications* [60]. Our book can be viewed as a companion text, with consistent notation, themes, and software packages. Students with this experience can move quickly through the applications, focusing on the traditional theory and its connections to discrete transformations. Students completing the discrete course should have a good sense of where the material is headed, as well as motivation to see the theoretical development of the various discrete transform filters. In this case, some sections of the text can be omitted and the entire book could be covered in one semester.

A third option exists for students who have a strong background in Fourier analysis. In this case, the instructor could concentrate heavily on the theoretical ideas in Chapters 5, 6, 8, and 9 and develop a real appreciation for how Fourier methods can be used to drive the theory of multiresolution analysis and filter design.

Problem Sets, Software Package, and Web Site

Problem solving is an essential part of learning mathematics, and we have tried to provide ample opportunities for the student to do so. After each section there are problem sets with a variety of exercises. Many allow students to fill in gaps in proofs from the text narrative, as well as to provide proofs similar to those given in the text. Others are fairly routine paper–pencil exercises to ensure that students understand examples, theorem statements, or algorithms. Many require computer work, as discussed in the next paragraph. We have provided 430 problems in the book to facilitate student comprehension material covered. Problems marked with a ★ should be assigned and address ideas that are used later in the text.

Many concepts in the book are better understood with the aid of computer visualization and computation. For these reasons, we have built the software package `ContinuousWavelets` to enhance student learning. This package is modeled after the `DiscreteWavelets` package that accompanies Van Fleet's book [60]. These packages are available for use with the computer algebra systems (CAS)

Mathematica®, Matlab®, and Maple™. This new package is used in the text to investigate a number of topics and to explore applications. Both packages contain modules for producing all the filters introduced in the course as well as discrete transformations and their inverses for use in applications. Visualization tools are also provided to help the reader better understand the results of transformations. Modules are provided for applications such as data compression, signal/image denoising, and image segmentation. The `ContinuousWavelets` package includes routines for constructing scaling functions (via the cascade algorithm) and wavelet functions. Finally, there are routines to easily implement the ideas from Chapter 3 — students can easily construct piecewise constant functions and produce nice graphs of projections into the various V_j and W_j spaces.

The course Web site is

`http://www.stthomas.edu/wavelets`

On this site, visitors will find the software packages described above, several computer labs and projects of varying difficulty, instructor notes on teaching from the text, and some solutions to problems.

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D.K.R. and P.V.F.

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