Discrete and Combinatorial Mathematics

An Applied Introduction

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ADDISON-WESLEY PUBLISHING COMPANY

READING, MASSACHUSETTS ▪ MENLO PARK, CALIFORNIA ▪ DON MILLS, ONTARIO ▪ WOKINGHAM, ENGLAND
AMSTERDAM ▪ SYDNEY ▪ SINGAPORE ▪ TOKYO ▪ MEXICO CITY ▪ BOGOTÁ ▪ SANTIAGO ▪ SAN JUAN

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Library of Congress Cataloging in Publication Data

Grimaldi, Ralph P.
Discrete and combinatorial mathematics.

Includes index.
1. Mathematics — 1961– . 2. Electronic data processing — Mathematics. 3. Combinatorial analysis.
I. Title.
QA39.2.G748 1985 510 84-9359
ISBN 0-201-12590-0

Reprinted with corrections, June 1985

BCDEFGHIJ-HA-898765

Discrete and Combinatorial Mathematics

DEDICATED TO
MY MOTHER AND FATHER

P R E F A C E

Resulting from the technological advances of the last two decades, changes in the undergraduate curriculum have included the introduction of courses in (1) combinatorics (or counting) and (2) discrete methods (that appeal to the *finite* nature of certain problems). One reason for this is the abundance of applications of these mathematical disciplines in computer science and operations research. Discrete methods and combinatorial reasoning underly the areas of data structures, as well as computational complexity and the analysis of algorithms. Consequently, many majors in computer science are also required to take courses in these disciplines. In addition to applications in computer science, one also finds applications in engineering and the physical and life sciences, and in statistics and the social sciences. Therefore the area of discrete mathematics provides valuable training for students in areas besides mathematics and computer science.

The objective of this book is to provide an introductory survey in both combinatorial and discrete mathematics. Since it is intended for the beginning student, there are a great number of examples with detailed explanations. (Each example is separately numbered and an open square is used to denote the end of each example.) In addition, wherever proofs are given, they too are presented with sufficient detail.

The text strives to accomplish the following goals:

1. To introduce the student at the sophomore-junior level, if not earlier, to the topics and techniques of combinatorial reasoning and discrete methods. Problems in counting, or enumeration, require a careful analysis of structure (e.g., whether or not order is important) and logical possibilities. There may even be a question of existence for some situations. Following such a careful analysis, we shall often find that the solution of a problem requires simple techniques for counting the possible outcomes that evolve from the breakdown of the given problem into smaller subproblems.
2. To introduce a wide variety of applications. In this regard, where structures from abstract algebra are required, only the basic theory needed for the application is developed. Furthermore, the solutions of some applications lend themselves to iterative procedures that lead to specific algorithms. The algorithmic approach to the solution of problems is fundamental in discrete mathematics. This approach reinforces the close ties between this discipline and the area of computer science.
3. To develop the mathematical maturity of the student through the study of an area that is so different from the calculus. Here, for example, there is the opportunity to establish results by counting a certain collection of objects in more than one way. This provides what are called combinatorial identities and introduces a novel proof technique. Proofs by mathematical induction are also used throughout the text, following their development in Chapter 6. Prior to Chapter 6 the ideas behind induction appear in certain recursive definitions and examples.

With regard to proofs in general, an attempt has been made to motivate theorems from observations on specific examples. In addition, whenever a finite situation provides a result that is not true for the infinite case, this situation is singled out for attention. Closed squares are used in the text to indicate the end of a proof. Proofs that are extremely long and/or rather special in nature are omitted. However, for the small number of proofs that are omitted, references are supplied for the reader interested in seeing the validation of these results. (The amount of emphasis placed on proofs will depend on the goals of the individual instructor and his or her student audience.)

4. To present an adequate survey of topics for the computer science student who will be taking more advanced courses in data structures and algorithm analysis. The coverage here on groups, rings, and fields will also provide an applied introduction for mathematics majors who wish to continue their study of abstract algebra.

The prerequisites needed for this book are primarily a sound background in high school algebra and an interest in attacking and solving a variety of problems. No particular programming ability is assumed. There are a few programs that appear in the text, but these are designed to reinforce particular examples. Such results may be skipped without any loss of continuity. With regard to calculus, we shall mention later in this preface its extent in Chapters 10 and 11.

My major motivation for writing this book is the encouragement I've received over the past seven years from my students and colleagues. This text reflects both my interests and those of my students, as well as the current recommendations of the Committee on the Undergraduate Program in Mathematics and the Association of Computing Machinery.

Since the areas of discrete and combinatorial mathematics are fairly new to the undergraduate curriculum there are many opinions as to which topics should be included in such courses. As each instructor and student will have different interests and needs, the coverage here is rather broad, as a survey course mandates. Yet there are many topics that some readers may feel should also be covered. Furthermore, there may be some differences of opinion with regard to the order in which the topics included here are presented. The order here rests upon my conviction that enumeration can reinforce the study of structure and vice versa.

Despite the interweaving of structure and enumeration, the chapters following Chapter 6 have been developed as independently as possible. The first six chapters form the underlying core of the text and provide enough material for a one-quarter or one-semester course. A second course that emphasizes combinatorics should include Chapters 7, 10, 11, and sections 1, 2, 3, 9, 10, 11 of Chapter 12. (In Chapter 10 some results from calculus are used; namely, the differentiation of algebraic functions, and partial fraction decompositions. For those who wish to skip this chapter, the first three sections of Chapter 11 can still be covered.) For a course in the role of discrete structures in computer science, the material in Chapters 9, 14, 15, 16, and sections 1–8 of Chapter 12 provides applications on switching functions and coding theory, and an introduction to graph theory and trees, and their role in optimization. Finally, a course in applied algebra can be developed after Chapter 6, and this should include coverage of Chapters 8, 9, 12, and 13. Other possible courses can be developed by consideration of the following dependency table.

Chapter	Dependence on Prior Chapters
1	No dependence
2	1
3	1, 2
4	1, 2, 3
5	1, 2, 3, 4
6	Minor dependence on 1, 2, 3
7	1, 2
8	3, 5, 6 (The Euler ϕ function is used here. This function is established in Section 7.1 but the result can be used in Chapter 8 without doing Chapter 7.)
9	2, 3, 5
10	1, 2
11	1, 2, 10
12	1, 2, 3, 5, 6
13	2, 3, 5, 6, 8
14	1, 2, 3, 6 (Although some graph theoretic ideas are mentioned in Chapters 4, 5, 7, and 9, this chapter is developed with no dependence on these earlier results.)
15	1, 2, 3, 6, 14
16	2, 14, 15.

In regard to the dependence of one section of a chapter on earlier sections, one should anticipate some dependence in the section exercises. Also, at the end of each chapter is a set of miscellaneous exercises where ideas from several chapters may be needed for the solutions. The overall role of the exercises is a key one. The exercises at the end of each section are designed to: (1) review the material in that section; (2) tie together ideas from earlier sections of the chapter; and (3) develop further concepts related to the material in the section. A few exercises call for computer programs to implement a given example or algorithm. These are designed for students with a minimal amount of programming experience. Answers are provided at the back of the text for almost all of the odd-numbered exercises.

In addition to the miscellaneous exercises, each chapter is concluded with a summary and historical review of the major ideas covered in that chapter. This should provide an overview of the development of the concepts in the chapter and provide information on further applications. A list of references for further reading also appears at the end of each chapter.

If space permitted, I would mention each of the students who took courses in discrete mathematics and combinatorics from me and suggested putting my class notes into a book. To those students who worked from the mimeographed version of this book I owe many thanks for finding mistakes and suggesting ways to improve the exposition. Most helpful in this category were Paul Griffith, Meredith Vannauker, Paul Barloon, Byron Bishop, Lee Beckham, Brett Hunsaker, Tom Vanderlaan, Michael Bryan, Charles Wilson, and Richard Nichols. I thank Lawrence Alldredge and Martin Rivers for reviewing several chapters of the text, and Lawrence Alldredge, Barry Farbrother, Paul Hogan, Dennis Lewis, and Charles Kyker for their enlightening comments on some of the programs and applications mentioned in the text. I gratefully acknowledge the persistent enthusiasm and optimism of my editors, Wayne Yuhasz and Jeff Pepper, as well as Mary Crittenden, Herb Merritt, and Maria Szmauz, among other members of the Addison-Wesley staff who assisted in the fulfillment of this project. The overall reviewers — Robert Crawford of Western Kentucky University, Carl Eckberg of San Diego State University, and especially Douglas Shier of Clemson University — deserve a special note of thanks for their very thorough work. I am also indebted to my colleagues, John Kinney, Gary Sherman, and especially Alfred Schmidt, for their encouragement throughout the two years spent on writing this book. I believe they are somewhat responsible for a great deal of what is of value here. However, if there is one person to whom I owe the greatest note of thanks, it is definitely the ever-patient and encouraging Mary Lou McCullough who typed and retyped and . . . to bring out the best in the manuscript. Alas, any remaining errors, ambiguities, or misleading results rest upon my shoulders alone.

Terre Haute, Indiana
December 1984

R.P.G.

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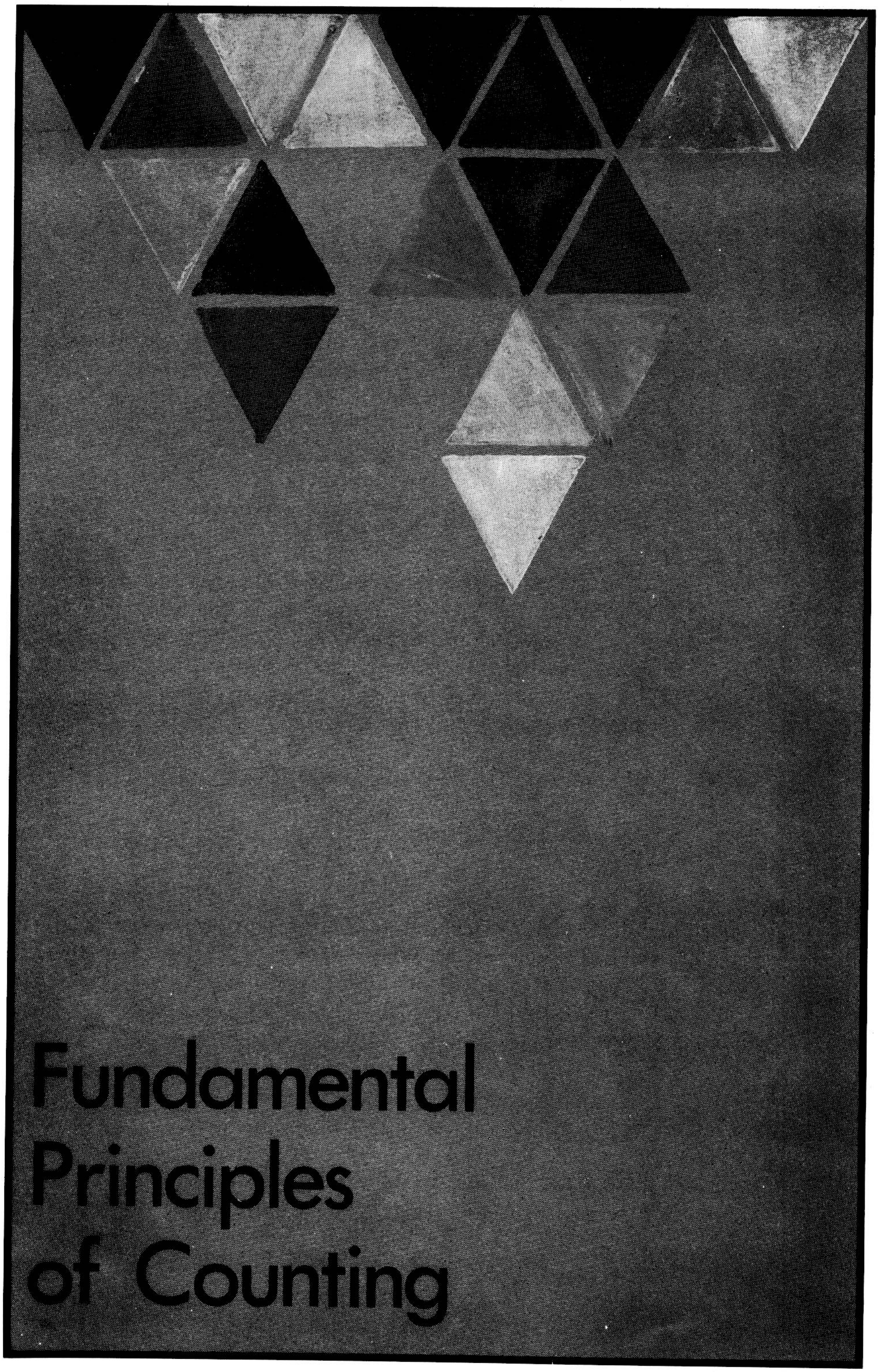

Fundamental Principles of Counting

Enumeration, or counting, may strike one as an obvious process that a student learns when first studying arithmetic. But then, it seems, that very little attention is given to any further development in counting as the student turns to "more difficult" areas in mathematics, such as algebra, geometry, trigonometry, and calculus. Consequently, this first chapter should provide some warning as to the seriousness and difficulty of "mere" counting. Enumeration does not end with arithmetic; there is quite a bit more to it. And as we enter this fascinating field of mathematics, we shall come upon many problems that are very simple to state but somewhat "sticky" to solve.

Beware of formulas! Without an analysis of each problem, a mere knowledge of formulas will prove next to useless. Instead, welcome the challenge to solve problems different from routine problems or past experiences. Seek solutions based on your own scrutiny, regardless of whether it follows what the author provides. There are often several ways to solve a given problem.

1.1 ■ THE RULES OF SUM AND PRODUCT

Our study of discrete and combinatorial mathematics begins with two basic principles of counting: the rules of sum and product. These appear quite simple in statement and initial application. In analyzing more complicated problems, one often is able to decompose such problems into parts that can be handled using these basic principles. We want to develop the ability to decompose such problems and piece together the partial solutions to arrive at the final answer. This will be done by analyzing and solving many diverse enumeration problems, with recognition of the principles being used in the solutions. Our first principle of counting can be stated as follows.

> **The Rule of Sum:** If a first task can be performed in m ways, while a second task can be performed in n ways, and the two tasks cannot be performed simultaneously, then performing either task can be accomplished in any of $m + n$ ways.

Before demonstrating its use, we state here an observation that will be true throughout the entire text: When we say that a particular occurrence, such as a first task, can come about in m ways, these m ways are assumed to be distinct, unless a statement is made to the contrary.

EXAMPLE 1.1 A college library has 40 textbooks on sociology and 50 textbooks dealing with anthropology. By the rule of sum a student at this college can select among $40 + 50 = 90$ textbooks in order to learn more about one of these two subjects. □

EXAMPLE 1.2 The rule can be extended beyond two tasks as long as no pair of them can occur simultaneously. For instance, a computer science instructor who has, say, five introductory books each on APL, BASIC, FORTRAN and Pascal can select any one of these 20 to give to a student interested in learning a first programming language. □

EXAMPLE 1.3 The computer science instructor of Example 1.2 has two colleagues. One of these colleagues has three textbooks on algorithm analysis, and the other has five such textbooks. If n denotes the number of books this instructor can borrow on this topic, then $5 \leq n \leq 8$, for here the colleagues may be in possession of copies of the same textbook. □

With the rule of sum taken care of we now consider the following example, which will introduce us to the rule of product.

EXAMPLE 1.4 In trying to reach a decision on plant expansion, an administrator assigns 12 of her employees to two committees. Committee A consists of five members and is to investigate possible favorable results from such an expansion. The other seven employees, committee B, will scrutinize possible unfavorable repercussions. Should the administrator decide to speak to just one committee member before making her decision, then by the rule of sum there are 12 employees she can call upon for input. However, in order to be a bit more unbiased, she decides to speak with a member of committee A on Monday and then a member of committee B on Tuesday, before reaching a decision. Using the following principle we find that she can speak with two such employees in $5 \times 7 = 35$ ways. □

The Rule of Product: If a procedure can be broken down into first and second stages, and if there are m possible outcomes for the first stage and n for the second stage, then the total procedure can be carried out, in the designated order, in mn ways.

This rule is sometimes referred to as the *principle of choice*.

EXAMPLE 1.5 The drama club of Central University is having tryouts for a spring play. With six men and eight women auditioning for the leading male and female roles, by the rule of product the director can cast his leading couple in $6 \times 8 = 48$ ways. □

EXAMPLE 1.6 Here various extensions of the rule are illustrated by considering the manufacture of license plates consisting of two letters followed by four digits.

a) If no letter or digit can be repeated, there are $26 \times 25 \times 10 \times 9 \times 8 \times 7 = 3,276,000$ different possible plates.
b) With repetitions of letters and digits allowed, we find $26 \times 26 \times 10 \times 10 \times 10 \times 10 = 6,760,000$ license plates possible.
c) If repetitions are allowed, how many of the plates in part (b) have both letters vowels (a, e, i, o, u) and all digits even? (0 is an even integer.) □

EXAMPLE 1.7 At times it is necessary to combine several different counting principles in the solution of one problem. Here we find that both the rules of sum and product are needed to attain the answer.