Classical Artinian Rings and Related Topics

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$$\begin{array}{ccccl} 0 \longrightarrow & X & \xrightarrow{i} & B = & B_1 \oplus B_2 \\ & f \downarrow & & & h_1 \downarrow \quad \uparrow h_2 \\ & A & & = & A_1 \oplus A_2 \\ & & & & \qquad\quad \uparrow \\ & & & & \qquad\quad 0 \end{array}$$

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$$\begin{pmatrix} Q^\star & Qc^2 & Qc^2 & Qc\alpha_{12} \\ Qc^2 & Q^\star & Qc^2 & Qc\alpha_{12} \\ Qc^2 & Qc^2 & Q & Q\alpha_{12} \\ Qc\beta_{21} & Qc\beta_{21} & Q\beta_{21} & Q \end{pmatrix} \Big/ \begin{pmatrix} 0 & Qc^3 & Qc^3 & Qc^2\alpha_{12} \\ Qc^3 & 0 & Qc^3 & Qc^2\alpha_{12} \\ Qc^3 & Qc^3 & 0 & 0 \\ Qc^2\beta_{21} & Qc^2\beta_{21} & 0 & 0 \end{pmatrix}$$

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Dedicated to Professor Manabu Harada

Preface

One century or more has passed since the study of noncommutative ring theory began with the pioneering work of its forefathers Wedderburn, Frobenius, Noether, Artin, von Neumann and many others. Since its birth, the theory has developed extensively, both widely and deeply. A survey of its history reveals the following key topics: the representation of finite groups, Frobenius algebras, quasi-Frobenius rings, von Neumann regular rings, homological algebra, category theory, Morita equivalence and duality, the Krull-Remak-Schmidt-Azumaya Theorem, classical quotient rings, maximal quotient rings, torsion theory, the representation of quivers, etc. In spite of its wide range, ring theory is roughly divided into four branches; namely noetherian rings, artinian rings, von Neumann regular rings and module theory with Wedderburn-Artin's structure theorem on semisimple rings located at the crossroads.

One of roots of the theory of artinian rings is the historical article [51] of Frobenius in 1903. In this paper, Frobenius studied finite dimensional algebras over a field whose left right regular representations are equivalent. In the late 1930s and early 1940s, these "Frobenius algebras" were reviewed and studied in Brauer-Nesbitt [24], Nesbitt [138], Nakayama-Nesbitt [136] and Nakayama [132], [133]. In particular, Nakayama developed these algebras by giving ring-theoretic characterizations and introduced Frobenius rings and quasi-Frobenius rings (QF-rings) in the eliminated forms from the dependency of the operation of the field. Thereafter QF-rings were continuously studied and became one of the central subjects in the study of ring theory.

Let us briefly retrace the progress of QF-rings. By the method of homological algebra, QF-rings have been studied by many peaple. In 1951, Ikeda [82] (cf. [83]) characterized QF-rings as the self-injective artinian rings, and in 1958, Morita published his famous paper [122] (cf. [123]), in which a QF-ring R was characterized as an artinian ring with the property that $\mathrm{Hom}_R(-, R)$ defines a duality between the category of finitely generated left R-modules and that of finitely generated right R-modules. Prior to these works, in 1948, Thrall generalized QF-rings by introducing *QF-1* rings, *QF-2* rings and *QF-3* rings in [174] (cf. Nesbitt-Thrall [139]). QF-rings are important in the study of representations of finite groups since each group algebra of a finite group over a field is QF. We emphasis that QF-rings are artinian rings which have an abundance of properties and a deep structure, but there are still several outstanding unsolved problems on these rings such as the Nakayama conjecture and the Faith conjecture, to name just a few.

In the same [133], Nakayama also introduced another artinian rings which are called generalized uniserial rings. These rings are just artinian serial rings, and are also called Nakayama rings in honor of Nakayama and in our book we use this latter terminology.

Comparing Nakayama rings with QF-rings, we are able to get a deep insight into the structure of Nakayama rings. In [134] (cf. Eisenbud-Griffith [43]), Nakayama showed that every module over a Nakayama ring can be expressed as a direct sum of uniserial modules. This fact shows that Nakayama rings are the most typical rings of finite representation type.

Nakayama rings appear in the study of non-commutative noetherian rings. Indeed, the Eisenbud-Griffith-Robson Theorem ([43], [44]) states that every proper factor ring of a hereditary noetherian prime ring is a Nakayama ring, therefore this classical theorem turns the spotlight on Nakayama rings. Kupisch made a major contribution in the study of Nakayama rings. He introduced what is now called the Kupisch series and, at the same time, was the first to discover skew matrix rings. As we will see in this book, skew matrix rings over local Nakayama rings are the essence of Nakayama rings.

In the current of the study of QF-rings, there are two features: one is to give characterizations of these rings and the other is to generalize these rings. However in this book we study these rings from different viewpoints by exhibiting several topics.

In the early 1980s, Harada found two new classes of artinian rings which contain QF-rings and Nakayama rings. These rings are introduced as mutually dual notions. Oshiro [147] studied these new ring classes, naming them Harada rings (H-rings) and co-Harada rings (*co-H-rings*) and showed the unexpected result that left H-rings and right co-H-rings coincide.

The main objective of this book is to present the structure of left H-rings and to apply this to artinian rings, including QF-rings and Nakayama rings, giving a new perspective on these classical artinian rings. In particular, several fundamental results on QF-rings and Nakayama rings are described, and a classification on Nakayama rings is exhibited. We hope that, through this book, the readers can come to appreciate left H-rings and gain a better understanding of QF-rings and Nakayama rings.

For the study of direct sums of indecomposable modules, Azumaya's contribution is inestimable. In the original Krull-Remak-Schmidt Theorem, the indecomposable modules were assumed to have both the ascending and descending chain conditions. However, in [10], Azumaya considered completely indecomposable modules, that is, modules with local endomorphism rings, replacing the chain conditions, and improved the Krull-Remak-Schmidt Theorem to what is now referred to as the Krull-Remak-Schmidt-Azumaya Theorem, a very useful tool in the study of module theory. The final version of the Krull-Remak-Schmidt-Azumaya Theorem was completed by Harada (see [64]) using the factor categories induced from completely indecomposable modules. It can be stated as follows: If M is a direct sum of completely indecomposable modules $\{M_\alpha\}_{\alpha \in I}$, then M satisfies the (finite) exchange property if and only if $\{M_\alpha\}_{\alpha \in I}$ is locally semi-T-nilpotent in the sense of Harada-Kanbara [71].

As one of the applications of his work on the Krull-Remak-Schmidt-Azumaya Theorem, in the late 1970s Harada introduced extending and lifting properties in certain classes of modules. Although these properties were initially rather specialized, these lead to their introduction in a more general setting in [147], [148] as extending modules and lifting modules. However, it should be noted that, in the early days of ring theory, the extending property explicitly appeared in Utumi's work [176]; in particular, in today's terminology, Utumi showed that a von Neumann regular ring R is right continuous if and only if the right R-module R is extending. Furthermore, he introduced the notion of a continuous ring by using the extending property. On the other hand, the lifting property implicitly appeared in Bass's work [21], where, in essence, it was proved that a ring R is semiperfect if and only if the right R-module R is lifting.

Generalizing the concept of continuous rings to module theory, continuous modules and quasi-continuous modules were introduced by Jeremy [87] in 1974 by using the extending property. This concept and its generalizations remained relatively dormant until Harada's work on lifting and extending properties appeared. Indeed, from the early 1980s, many ring theorists began to get interested in the study of the extending and lifting properties and, during the last 30 years, this field has developed at an incredible rate. This situation is witnessed in the following books on this field: M. Harada, *Factor Categories with Applications to Direct Decomposition of Modules* (Marcel Dekker, 1983); Mohamed-Müller, *Continuous Modules and Discrete Modules* (Cambridge Univ. Press, 1990); Dung-Huynh-Smith-Wisbauer *Extending Modules* (Pitman, 1994); and Clark-Lomp-Vanaja-Wisbauer, *Lifting modules* (Birkhäuser, 2007).

Although Harada introduced his classes of H-rings and *co*-H-rings prior to his study of lifting and extending properties, it seems to be unclear whether he was conscious of the relationships between these properties and rings. However, in [147], [148], module-theoretic characterizations of left H-rings and Nakayama rings were given using the properties. Indeed, the results obtained in these papers lead us to claim that the extending and lifting properties really have their origin in the theory of artinian rings and so, in this sense, these concepts have been studied extensively.

In Chapter 1, we provide a background sketch of fundamental concepts and well-known facts on artinian rings and related materials. In particular, since materials in this book is based on semiperfect rings, we give known facts on these rings by using the lifting property of modules. Basic QF-rings, Nakayama permutations and Nakayama automorphisms as rings are also considered here for later use.

In Chapter 2, we present Fuller's Theorem which provides a criterion for the injectivity of an indecomposable projective module over one-sided artinian rings. We present an improved version of Fuller's Theorem for semiprimary rings by Baba and Oshiro [20], and introduce a more general presentation due to Baba [11].

In Chapter 3, we introduce one-sided H-rings and one-sided *co*-H-rings and give module-theoretic characterizations of these rings, and show that left H-rings and right *co*-H-rings coincide. Among the several characterizations of left H-rings, we show that they are precisely the rings R for which the class of projective right R-modules is closed under taking essential extensions.

In Chapter 4, fundamental structure theorems of H-rings are explored. We establish the matrix representation of left H-rings. It is shown that for a given basic indecomposable left H-ring R, there exists a QF-subring $F(R)$ of R, from which R can be constructed as an upper staircase factor ring of a block extension of $F(R)$. Because of this feature, we say that $F(R)$ is the *frame* QF-subring of R.

In Chapter 5, we study self-duality of left H-rings. For a basic indecomposable QF-ring F, it is shown that every basic indecomposable left H-ring R with $F(R) \cong F$ is self-dual if and only if F has a Nakayama automorphism. In Koike [98], it is noted that an example in Kraemer [103] provides a basic indecomposable QF-ring without a Nakayama automorphism. The difference between QF-rings with or without Nakayama automorphisms affects self-duality of their factor rings and block extensions.

In Chapter 6, we introduce skew matrix rings and develop fundamental properties of these rings. As mentioned earlier, these rings were introduced by Kupisch [105] and independently by Oshiro [149] through the study of Nakayama rings. In this book, we use the definition of skew matrix rings given in the latter paper. We will illustrate the usefulness of skew matrix rings for the study on QF-rings and Nakayama rings.

In Chapter 7, we develop a classification of Nakayama rings as an application of left H-rings. We study basic indecomposable Nakayama rings by analyzing certain types of Nakayama permutations of their frame QF-subrings. It is shown that a basic indecomposable Nakayama rings is represented as an upper staircase factor ring of a block extension of the frame QF-subring and, in particular, every basic indecomposable Nakayama ring whose frame QF-subring is not weakly symmetric can be directly represented as an upper staircase factor ring of a skew matrix ring over a local Nakayama ring. As an application of left H-rings, we confirm the self-duality of Nakayama rings by applying the fact that skew matrix rings over local Nakayama rings have Nakayama automorphisms.

In Chapter 8, we give several module-theoretic characterizations of Nakayama rings by using extending and lifting properties. These characterizations provide relationships between QF-rings, Nakayama rings and one-sided H-rings.

In Chapter 9, we study Nakayama algebras over an algebraically closed field. It is shown that such algebras are represented as factor rings of skew matrix rings over one variable polynomial rings over algebraically closed fields. This result inspires us to study Nakayama group algebras over such fields. We summarize several results on these group algebras in relationships to skew matrix rings.

In Chapter 10, we introduce the Faith conjecture, and as a by-product of the study of this conjecture, we give a new way of constructing local QF-rings with Jacobson radical cubed zero. This construction produces many local QF-rings which are not finite dimensional algebras.

We end the text with several questions on QF-rings and Nakayama rings.

Though we have tried to make this book as self-contained as possible, for basic notions and well-known facts on non-commutative ring theory (such as semiprimary rings, semiperfect rings, quasi-Frobenius rings, and Morita duality etc.), the reader is referred to the books Faith [46], [47], Anderson-Fuller [5], Harada [64], Lam [107], and Wisbauer [182], and, in addition, to the books Dung-Huynh-Smith-Wisbauer [41] for extending modules, Puninski [162] for serial rings, Xue [184] for Morita duality, and Nicholson-Yousif [142] and Tachikawa [172] for QF-rings. In particular, for the background of Frobenius algebras and QF-rings, the reader is referred to Faith [49], Lam [107], Nakayama-Azumaya [137], Nicholson-Yousif [142], and Yamagata [189].

We are indebted to a large number of people for this overall effort possible. In particular we are thankful to John Clark, Jiro Kado, Kazutoshi Koike as well as Akihide Hanaki, Shigeo Koshitani, Isao Kikumasa, and Hiroshi Yoshimura for much of their help. Thanks also to Derya Keskin, Yosuke Kuratomi, Cosmin Roman and Masahiko Uhara, and Kota Yamaura for their comments which improved the overall presentation.

List of Symbols

R : an associative ring with $1 \neq 0$.
M_R : a unitary right module over a ring R.
${}_R M$: a unitary left module over a ring R.
$Pi(R)$: a complete set of orthogonal primitive idempotents of a semiperfect ring R.
$Z(M)$: the singular submodule of a module M.
id_X or, simply, id if no confusion arises : the identity map of X.
$J(M)$: the Jacobson radical of a module M.
$J_i(M)$: the i-th radical of a module M.
$J(R)$ or, simply, J : the Jacobson radical of a ring R.
$S(M)$: the socle of a module M.
$S_k(M)$: the k-th socle $\{ x \in M \mid xJ^k = 0 \text{ (resp. } J^k x = 0) \}$ of a right (resp. left) R-module M.
$T(M)$: the top $M/J(M)$ of M.
$E(M)$: an injective hull of a module M.
$L(M)$: the Loewy length of a module M.
$| M |$: the composition length of a module M.
$N \subseteq_e M$: N is an essential submodule of a module M.
$N \subseteq_c M$: N is a co-essential submodule of a module M.
$N \ll M$: N is a small (superfluous) submodule of a module M.
$N \underset{\sim}{\subseteq} M$: N is isomorphic to a submodule of M.
$(a)_L : S \longrightarrow aS$: the left multiplication map by a, where S is a subset of a ring R and $a \in R$.
$(a)_R : S \longrightarrow Sa$: the right multiplication map by a, where S is a subset of a ring R and $a \in R$.
$r_S(T)$: the right annihilator of T in S.
$l_S(T)$: the left annihilator of T in S.
$\#S$: the cardinal of a set S.

$\mathbb{N}$: the set of positive integers.
$\mathbb{N}_0$: the set of non-negative integers.
$M^{(I)}$: a direct sum of $\#I$ copies of a module M.
M^I : a direct product of $\#I$ copies of a module M.
$R^{(\mathbb{N})}$: a direct sum of countable copies of a ring R.
Mod-R (resp. *R-Mod*) : the category of all right (resp. left) R-modules.
FMod-R (resp. *R-FMod*) : the category of all finitely generated right (resp. left) R-modules.
$(Q)_{\sigma,c,n}$: the skew matrix ring over Q with respect to (σ, c, n).
$\{\alpha_{ij}\}$, $\{\alpha_{ij} \mid i < j\} \cup \{\beta_{ij} \mid i < j\}$: the skew matrix units.
$\langle a \rangle_{ij}$: the matrix with (i,j)-entry 0 and other entries 0.
$\langle X \rangle_{ij}$: the set $\{\langle x \rangle_{ij} \mid x \in X\}$.
$[i]$: the least positive residue modulo m.
$F(R)$: the frame *QF*-subring of R.
$\dim(V_D)$: the dimension of a right vector space V_D over the division ring D.
$A \subsetneqq B$: A is isomorphic to a submodule of B.

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Contents