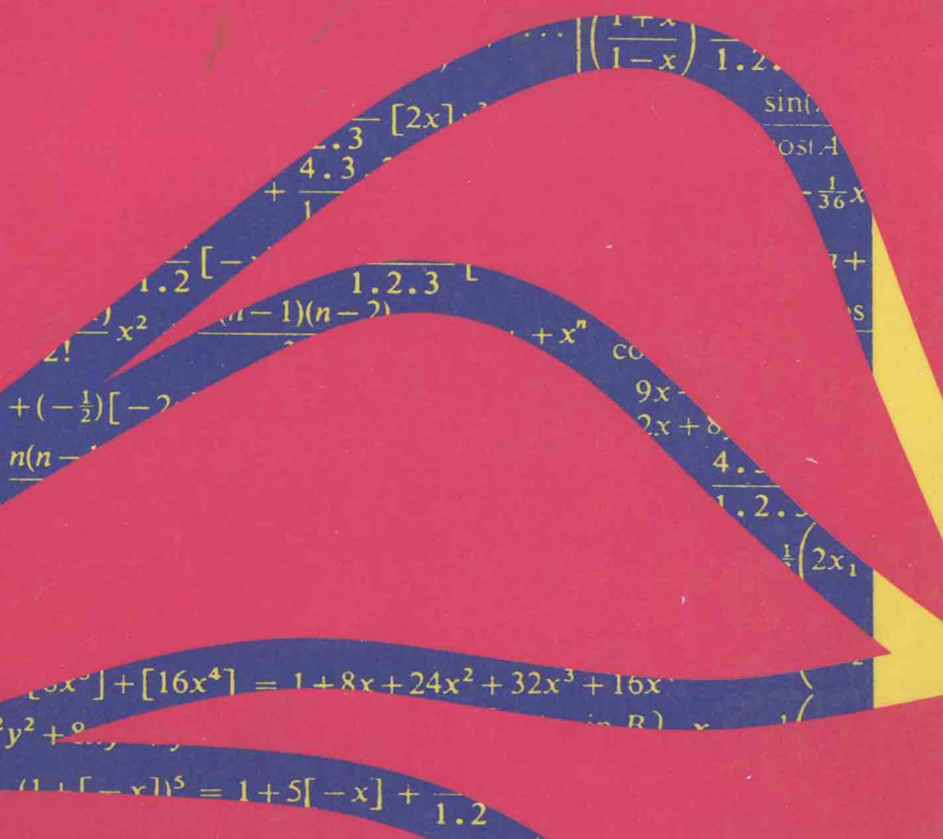
MATHEMATICS FOR ECONOMISTS

A FIRST COURSE

J. M. Pearson



Mathematics for economists

A first course

J. M. Pearson

Longman
London and New York

Longman Group Limited
Longman House
Burnt Mill, Harlow, Essex, UK

Published in the United States of America
by Longman Inc., New York

First published 1982

British Library Cataloguing in Publication Data

Pearson, J.M.
Mathematics for economists.
1. Economics, Mathematical
I. Title
510'.2433 HB135

ISBN 0-582-29615-3

Library of Congress Cataloguing in Publication Data

Pearson, J.M., 1946-
Mathematics for economists.

Bibliography: p.
Includes index.
1. Economics, Mathematical. I. Title.
HB135. P42 510'.24339 81-14309
ISBN 0-582-29615-3 AACR2

Printed in Singapore by Selector Printing Co Pte Ltd

Preface

This book is an introductory text in mathematics, designed primarily for undergraduate students of economics, although social science students, required to undertake at least some mathematical training as part of their degree, should also find it useful.

Economists use mathematics extensively to assist them in analysing economic problems, and although some economists lament the increasing use of some of the more high-powered mathematical techniques, which they claim obscure the real economics, it is a fact that mathematical techniques of the type introduced in this book provide an invaluable aid to the understanding and analysis of many economic problems. Consider, for example, a firm faced with deciding how much should be produced to maximise its profit. This is a key problem in economics, which all economists must understand. In order to analyse and solve problems posed by maximisation, the techniques of differential calculus are used. Thus it is only by having a good grasp of mathematics that an economist can successfully master economics. This fact is reflected in the increasing introduction of mathematical and quantitative methods into economic and social science courses.

This book provides a one-term course in mathematics for economists, for students who have progressed only as far as O-level mathematics. Since, in many cases, this was several years ago, the book revises topics covered in an O-level syllabus (sets, graphs, equations etc.) and then introduces other topics students need to grasp in order to cope with an economics degree (differentiation, optimisation, integration). The book is particularly suitable for students possessing only O-level mathematics, who are taking a first-level undergraduate 'quantitative methods' course, consisting of half mathematics and half statistics. These students should find that most of the mathematics in their course is covered in this book. The book will also prove useful to those students, with only O-level mathematics, who are taking a full year's first level mathematics course, but find difficulty, at the beginning of the course, in coping with the texts recommended. This book will provide a 'bridge' between O-level (much of which needs revising) and these more advanced texts.

The book adopts the 'set' or 'modern mathematics' approach, although it covers the same material as in more traditional or 'old mathematics' textbooks. There are two reasons for this approach: firstly because set theory is increasingly being used as a technique in analysing economic problems, and secondly because it provides a unifying approach to the topics covered.

In addition to presenting mathematical techniques, the book attempts to demonstrate some of the applications of these techniques to elementary economic problems, by including at the end of some of the chapters, a section on 'Applications to economics'. These sections illustrate the relevance to economics of the mathematical techniques introduced in the chapter. For example in Chapter 2, where functions are introduced, the section on applications at the end of that chapter discusses demand and supply functions, cost, revenue and profit functions. In Chapter 6, where maximisation and minimisation of functions is covered, the 'Applications to economics' section discusses profit maximisation and the relation between marginal revenue and marginal cost, illustrating how a grasp of the mathematical techniques is invaluable in understanding the economics.

Chapter 1, on sets, does not include a section on 'Applications to economics', not because sets have no applications or uses in economics, but because the groundwork necessary to present useful applications requires more space than could be allocated in this book. (The reader interested in the applicability of sets to economic problems is referred to V. C. Walsh *Introduction to Contemporary Microeconomics*.) Although Chapter 1 would have been included in its own right, as an introduction to the ideas presented in, for example, Walsh, the main reason for its inclusion is that it provides both a foundation and a framework on which the other mathematical techniques can be discussed. As A. C. Chiang states '. . . . the concept of sets underlies every branch of modern mathematics', (A. C. Chiang, *Fundamental Methods of Mathematical Economics*), and it is in this belief that I have included the chapter and used the ideas developed in the chapter as the basis for the techniques discussed in subsequent chapters.

The book has several features which should make it attractive to many students.

Firstly, in keeping with recent development in mathematics teaching in the schools, it begins by introducing sets, a topic most people who have been taught mathematics in school in the last decade are familiar with. Many of the older mathematics for economists books do not mention the topic, and indeed many lecturers in the subject do not cover it, which seems strange to a generation of students nurtured on 'new Maths'.

Secondly, it introduces the mathematical topics as pure mathematics. Many students beginning an introduction to mathematics for economists course will not yet have learned any economics, and will not, initially, be able to apply the mathematical techniques they have learned. These students will be able to use the book leaving out the 'Applications to economics' sections. When they have learned some economics they will be able to go over the ground again, making use of the applications section. On the other hand, many students are reluctant to apply themselves to mastering the mathematical techniques, unless they can see the relevance of doing so. The applications sections are particularly useful for these students, as they can see immediately the relevance of the mathematical topics to economics. The application sections will also be particularly useful to those students who start the course already having some knowledge of economics.

Finally, it is hoped that this book will be especially useful to those students, the majority in many universities and polytechnics, who find it difficult to master mathematical techniques. The book covers every topic thoroughly, explains each point fully and clearly, and moves forward at a pace most students find comfortable. It illustrates points with graphs and examples wherever necessary and provides plenty of exercises for students to try out the techniques for themselves, and consolidate their grasp of the subject by working through problems.

My grateful thanks are extended to George Zis and Professor Michael Sumner, of Salford University and to Jon Stewart of Manchester University, for their encouragement, without which the book would not have been written, and their comments, without which it would have been an inferior publication. Thanks are also due to Mrs M. Ward, Mrs J. M. Robertson and Mrs B. Masters for their typing skills, and finally to my wife, not only for her help in typing and proofreading, but also for the encouragement and comfort she offered throughout the writing of the book.

J. M. Pearson

Contents

Chapter 1

Sets

In this chapter we lay the foundations for most of the mathematics which follows in subsequent chapters. We shall not provide economic examples of the use of set theory because of limitations of space. However, students will appreciate the value of the material in this chapter when they study subsequent chapters and those students interested in the economic applications of set theory are referred to *Introduction to Contemporary Microeconomics* by V. C. Walsh.

1.1 Introduction

We are all familiar with the use of the word 'set' in everyday language, but there is no harm in our having a formal definition.

Definition
A *set* is a collection of distinct objects.

For example, we can talk about the set of chairs in a room or the set of people over 65 years of age, or the set of planets in the solar system, or (one of my favourite sets) the set of female students with blue eyes. We shall denote sets by capital letters A, B, S, X, etc.

There are two ways of describing a set; by *enumeration* and by *description*.

If we let A represent the set of days of the week we can write, by *enumeration*

A = {Monday, Tuesday, Wednesday, Thursday, Friday, Saturday, Sunday} i.e. we list all the members of the set.

Sometimes it may be more convenient to use a *description* of a set.

Example
$A = \{a|a$ is a day of the week$\}$
which we read as 'A is the set of all (little) a such that (little) a is a day of the week'.
Notice that the 'common property' that the members of the set share, is included after the vertical bar.
Example
If P is the set of planets in the solar system, we can write, by *enumeration*
P = {Mercury, Venus, Earth, Mars, Jupiter, Saturn, Uranus, Neptune, Pluto}
or by *description*
$P = \{p|p$ is a planet in the solar system$\}$.

Clearly the more members there are in the set the more advantageous the second method becomes. Indeed, for some sets it may be impossible to use the method of enumeration.
Example
The set $M = \{m|m$ is a male living in Great Britain$\}$ could not easily be described by enumeration.

It is convenient at this point to introduce another definition and some more notation.

Definition
An object belonging to a set is called an *element* of the set.
Example
For our previous set
$M = \{m|m$ is a male living in Great Britain$\}$
John Brown is an element of M, which we write as
John Brown $\in M$
We can also write Jane Brown $\notin M$, i.e. Jane Brown does *not* belong to M.

Before proceeding further with our ideas on sets, it will be useful to introduce the 'real number system' as it is called.

1.2 The real number system

Definitions
The numbers we use for counting, e.g. 1, 2, 3, 4, etc. are called *positive integers*. (They are sometimes called natural numbers.)

The numbers -1, -2, -3, -4, etc. are similarly referred to as *negative integers*.
The set of numbers consisting of all the positive integers and the negative integers, with zero included for good measure, is called the set of *integers*.
We use the letter Z to denote the set of integers. So we can write
$Z = \{z | z \text{ is an integer}\}$
and $2 \in Z$

$-150 \in Z$

$0 \in Z$

We then have the set of *fractions*, for example $\frac{3}{4}$, $\frac{7}{8}$, $-\frac{1}{2}$. These together with the integers form the set of *rational* numbers.
We can use the letter Q to represent this set.
So we have $Q = \{q | q \text{ is a rational number}\}$
and $\frac{1}{2} \in Q$

$-\frac{111}{120} \in Q$

$2 \in Q$

Remark
Rationals are so-called because they are formed by taking the 'ratio' of two integers. Even the number 2 can be thought of as a ratio $\frac{2}{1}$.

Then there are some numbers, like, for example, $\sqrt{2}$ and the number π, which cannot be expressed as one integer over another. These numbers are called the set of *irrational* numbers. There are, in fact, a great many of these irrational numbers but in this book most of the numbers we meet will be from the set of rationals.

If we take all the rational numbers and irrational numbers together to form a set we get the set of *real* numbers. We represent the set of real numbers by the letter R.
So we have $R = \{r | r \text{ is a real number}\}$

$1 \in R$

$-7 \in R$

$\frac{1}{20} \in R$

$\sqrt{2} \in R$

$0 \in R$

N.B. It would be impossible to describe any of the sets introduced in this section by enumeration. They contain an *infinite* number of elements, i.e. the list is never-ending.

You might think that the set R contains every number we could ever wish to consider, but there are in fact millions of other numbers, somewhat esoteric in nature, but which do prove useful to mathematicians. These are called *unreal* or *imaginary* numbers. They occur when we try to take the square root of a negative number. Discussion of these is left until Chapter 4.

1.3 More about sets

In this section we shall introduce more definitions which will extend our ideas on sets.

Definition
Two sets A and B are said to be *equal* if they contain *exactly* the same elements.

This may seem a bit pedantic but we shall find we use the concept extensively later in the chapter and it is useful to have the definition formally stated.

Example
If $A = \{1, 2, 4\}$ and $B = \{2, 4, 1\}$
then $A = B$

N.B. The *order* in which the elements are written down does not matter with sets. The set consists of the numbers, *not* of the numbers in a specific order.

Example
If $A = \{a | a \text{ is a colour of the rainbow}\}$
$B = \{b | b \text{ is a primary colour}\}$
then $A \neq B$ because, for example, 'orange' is an element of A but not of B. So A and B do not contain exactly the same elements.

Definition
Suppose we have *two* sets X and Y. X is called a *subset* of Y, if every element of X is also an element of Y: i.e. if X is contained in Y, which we write as $X \subset Y$.

Example
If $X = \{1, 2, 3\}$ and $Y = \{1, 2, 3, 4\}$
X is a subset of Y, i.e. $X \subset Y$.

Example
If $A = \{a|a$ is a colour of the rainbow$\}$
$B = \{b|b$ is a primary colour$\}$
then B is a subset of A, i.e. $B \subset A$.
Example
If $Z = \{z|z$ is an integer$\}$
$R = \{r|r$ is a real number$\}$
then Z is a subset of R, i.e. $Z \subset R$.

Remark
If there is just *one* element of X which is *not* also an element of Y, then X is *not* a subset of Y. In fact, the easiest way to demonstrate that a set X is *not* a subset of Y is to find such an element.
Example
$X = \{-1, 1, 2\}$
$Y = \{y|y$ is a positive integer$\}$
X is *not* a subset of Y since $-1 \in X$ but $-1 \notin Y$.

Remark
A set is a subset of itself
Example
$Y = \{1, 2, 3, 4\}$
Y is a subset of Y according to our definition, i.e. $Y \subset Y$.

It is convenient to introduce a rather strange set at this point, the *empty set*.

Definition
The empty set is the set which contains *no* elements. It is denoted by the symbol $\varnothing$.

The empty set is considered to be a subset of *every* other set: i.e. $\varnothing \subset A$ for *any* set A.
(If $\varnothing$ were not a subset of any set A, it would be possible to find an element of $\varnothing$ which was not an element of A. However, $\varnothing$ has no elements at all so it is clearly impossible to find such an element.)

Remark
As we shall see later in this chapter, the empty set performs a similar function to that of zero in the real number system. However, we should not confuse $\varnothing$ with $\{0\}$. $\varnothing$ contains no

elements, whereas $\{0\}$ does contain an element, namely the number 0.

1.4 Operations on sets

We are all familiar with the idea that given any two *numbers* we can combine them to produce a third number using the operations of addition, subtraction, multiplication or division.

For example, given the numbers 5 and 8 we can combine them by addition to produce a third number 13: i.e. $5 + 8 = 13$.
Using a different operation, say multiplication, we get a different number: $5 \times 8 = 40$.

We shall now define operations on *sets*, i.e. ways of combining two sets to produce a third. There are two operations we shall be concerned with – the *union* and the *intersection*.

Definition
Given two sets, A and B, the *union* of these sets, is a set which contains those elements which belong to either A, or B or both.

When combining 5 and 8 by addition we can write $5 + 8$.
Similarly, when combining A and B by taking the union we can write $A \cup B$.
Example
If $A = \{1, 2, 4, 5, 6\}$ and $B = \{1, 3, 4, 9\}$
then $A \cup B = \{1, 2, 3, 4, 5, 6, 9\}$
N.B. If an element appears in both A and B, it is included in $A \cup B$, but only once.
Example
If $M = \{m|m$ is a male human being living in Great Britain$\}$
and $F = \{f|f$ is a female human being living in Great Britain$\}$
then $M \cup F = \{p|p$ is a human being living in Great Britain$\}$.
Example
If A is *any* set then
$A \cup A = A$
Example
If A is *any* set then
$A \cup \varnothing = A$.

The second operation on sets is called *intersection*.

Definition
Given two sets, A and B, the *intersection* of these sets is a set which contains only those elements which belong to both A and B.
We write the intersection of A and B as $A \cap B$. The intersection then contains elements *common* to both sets.
Example
If $A = \{1, 2, 4, 5, 6\}$ and $B = \{1, 3, 4, 9\}$
then $A \cap B = \{1, 4\}$.
Example
For *any* set A
$A \cap A = A$.
Example
For *any* set A
$A \cap \varnothing = \varnothing$
Example
If $M = \{m | m$ is a male human being living in Great Britain$\}$
$F = \{f | f$ is a female human being living in Great Britain$\}$
$M \cap F = \varnothing$, i.e. there is no element which is in M and also in F.
There are no common elements.

Definition
When two sets A and B are such that they have no common elements: i.e. $A \cap B = \varnothing$, we say they are *mutually exclusive* or *disjoint*.

We can naturally extend our operations to combine more than two sets. Referring back to our numbers again we can combine 5 and 8 by addition (to give 13) and then combine this with another number (say 4) to produce 17: i.e. $(5 + 8) + 4 = 17$.
The brackets are used in mathematics to tell us that we should combine 5 and 8 before adding 4. (We always work out the expression in the brackets first.) Similarly we have
$(5 \times 8) \times 4 = 160$.
(Because $5 \times 8 = 40$ and if we then multiply this by 4, we get 160.)

These ideas apply equally well to sets.
Example
If $A = \{1, 2, 4, 5, 6\}$ and $B = \{1, 3, 4, 9\}$
and $C = \{4, 7, 10\}$

then $A \cup B = \{1, 2, 3, 4, 5, 6, 9\}$
and $(A \cup B) \cup C = \{1, 2, 3, 4, 5, 6, 7, 9, 10\}$.

Example

If $A = \{a, b, c\}$ $B = \{b, e\}$ $C = \{d, e, g\}$
then $(A \cup B) = \{a, b, c, e\}$
and $(A \cup B) \cup C = \{a, b, c, d, e, g\}$.

Example

If $A = \{1, 2, 4, 5, 6\}$ $B = \{1, 3, 4, 9\}$ $C = \{4, 7, 10\}$
then $A \cap B = \{1, 4\}$
and $(A \cap B) \cap C = \{4\}$.

Example

If $A = \{a, b, c\}$ $B = \{b, e\}$ $C = \{d, e, g\}$
then $A \cap B = \{b\}$.
$(A \cap B) \cap C = \emptyset$ because $A \cap B$ and C have no elements in common.

We can use our ideas on numbers further. We said $(5 + 8) + 4$ meant combining 5 and 8 first, and then adding 4 to give 17. In fact, if we combine 8 and 4 first, and then add 5 we still get 17: i.e.

$$(5 + 8) + 4 = 17 \quad \text{and} \quad 5 + (8 + 4) = 17$$

So $(5 + 8) + 4 = 5 + (8 + 4)$.
So the brackets are not, in fact, needed here, and we just write $5 + 8 + 4 = 17$. Similarly $(5 \times 8) \times 4 = 160$ and $5 \times (8 \times 4) = 160$ so we don't bother with the brackets and we write $5 \times 8 \times 4 = 160$.

If we refer back to our sets we find that this is also true of them: i.e. $(A \cup B) \cup C$ gives exactly the same set as $A \cup (B \cup C)$. (The reader might try this out on the examples above.) So we can ignore the brackets here and write $A \cup B \cup C$. Similarly $(A \cap B) \cap C$ produces the same set as $A \cap (B \cap C)$ and we write $A \cap B \cap C$.
($A \cap B \cap C$ is the set which contains elements common to all three sets.)

The operations on numbers can give us further insight into those on sets.
Suppose we combine 5 and 8 by *addition* to give 13 and then combine this number with 4 by *multiplication* to give 52: i.e. we *mix* our operations.
Then we have $(8 + 5) \times 4 = 52$.