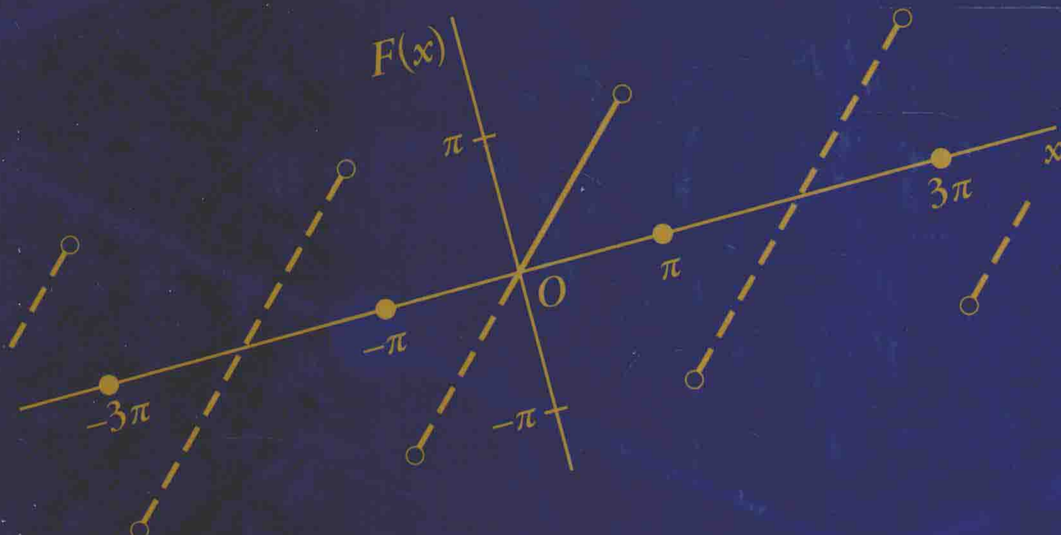
Eighth Edition

Fourier Series
and Boundary Value Problems



James Ward Brown
Ruel V. Churchill

$$F(x) = x = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx \quad (0 < x < \pi)$$

FOURIER SERIES AND BOUNDARY VALUE PROBLEMS

Eighth Edition

James Ward Brown

Professor of Mathematics
The University of Michigan–Dearborn

Ruel V. Churchill

Late Professor of Mathematics
The University of Michigan

Mc Graw Hill Connect Learn Succeed™

The McGraw-Hill Companies

Connect Learn Succeed™

FOURIER SERIES AND BOUNDARY VALUE PROBLEMS, EIGHTH EDITION

Published by McGraw-Hill, a business unit of The McGraw-Hill Companies, Inc., 1221 Avenue of the Americas, New York, NY 10020.

This book is printed on acid-free paper.

1 2 3 4 5 6 7 8 9 0 DOC/DOC 1 0 9 8 7 6 5 4 3 2 1

ISBN 978-0-07-803597-5
MHID 0-07-803597-X

Vice President & Editor-in-Chief: *Marty Lange*
Vice President, EDP/Central Publishing Services: *Kimberly Meriwether David*
Global Publisher: *Raghothaman Srinivasan*
Sponsoring Editor: *Bill Stenquist*
Marketing Director: *Thomas Timp*
Development Editor: *Lorraine Buczek*
Project Manager: *Melissa M. Leick*
Design Coordinator: *Margarite Reynolds*
Cover Designer: *Studio Montage, St. Louis, Missouri*
Cover Art: *Jenny Lindeman, Studio Montage*
Buyer: *Nicole Baumgartner*
Compositor: *MPS Limited, a Macmillan Company*
Typeface: *10/12 Times-Ten-Roman*
Printer: *R. R. Donnelley*

Library of Congress Cataloging-in-Publication Data

Brown, James Ward.
Fourier series and boundary value problems / James Ward Brown, Ruel V. Churchill.—8th ed.
p. cm.
Includes bibliographical references and index.
ISBN 978-0-07-803597-5
1. Fourier series. 2. Functions, Orthogonal. 3. Boundary value problems. I. Churchill, Ruel V. (Ruel Vance), 1899–1987. II. Title.
QA404.B76 2011
515'.2433—dc22 2010032734

www.mhhe.com

Brown and Churchill Series

Complex Variables and Applications, 8th Edition
Fourier Series and Boundary Value Problems, 8th Edition

The Walter Rudin Student Series in Advanced Mathematics

Bóna, Miklós: *Introduction to Enumerative Combinatorics*
Chartrand, Gary and Ping Zhang: *Introduction to Graph Theory*
Davis, Sheldon: *Topology*
Rudin, Walter: *Principles of Mathematical Analysis*
Rudin, Walter: *Real and Complex Analysis*

Other McGraw-Hill Titles in Higher Mathematics

Ahlfors, Lars: *Complex Analysis*
Burton, David M.: *Elementary Number Theory*
Burton, David M.: *The History of Mathematics: An Introduction*
Hvidsten, Michael: *Geometry with Geometry Explorer*

ABOUT THE AUTHORS

JAMES WARD BROWN is Professor of Mathematics at The University of Michigan–Dearborn. He earned his A.B. in physics from Harvard University and his A.M. and Ph.D. in mathematics from The University of Michigan in Ann Arbor, where he was an Institute of Science and Technology Predoctoral Fellow. He is coauthor with Dr. Churchill of *Complex Variables and Applications,* now in its eighth edition. He has received a research grant from the National Science Foundation as well as a Distinguished Faculty Award from the Michigan Association of Governing Boards of Colleges and Universities. Dr. Brown is listed in *Who's Who in the World*.

RUEL V. CHURCHILL was, at the time of his death in 1987, Professor Emeritus of Mathematics at The University of Michigan, where he began teaching in 1922. He received his B.S. in physics from the University of Chicago and his M.S. in physics and Ph.D. in mathematics from The University of Michigan. He was coauthor with Dr. Brown of *Complex Variables and Applications,* a classic text that he first wrote over 60 years ago. He was also the author of *Operational Mathematics*. Dr. Churchill held various offices in the Mathematical Association of America and in other mathematical societies and councils.

To the Memory of My Father,
George H. Brown,

and of My Long-Time Friend and Coauthor,
Ruel V. Churchill.

These Distinguished Men of Science for Years Influenced
The Careers of Many People, Including Mine.
J.W.B.

Joseph Fourier

JOSEPH FOURIER

JEAN BAPTISTE JOSEPH FOURIER was born in Auxerre, about 100 miles south of Paris, on March 21, 1768. His fame is based on his mathematical theory of heat conduction, a theory involving expansions of arbitrary functions in certain types of trigonometric series. Although such expansions had been investigated earlier, they bear his name because of his major contributions. Fourier series are now fundamental tools in science, and this book is an introduction to their theory and applications.

Fourier's life was varied and difficult at times. Orphaned by the age of 9, he became interested in mathematics at a military school run by the Benedictines in Auxerre. He was an active supporter of the Revolution and narrowly escaped imprisonment and execution on more than one occasion. After the Revolution, Fourier accompanied Napoleon to Egypt in order to set up an educational institution in the newly conquered territory. Shortly after the French withdrew in 1801, Napoleon appointed Fourier prefect of a department in southern France with headquarters in Grenoble.

It was in Grenoble that Fourier did his most important scientific work. Since his professional life was almost equally divided between politics and science and since it was intimately geared to the Revolution and Napoleon, his advancement of the frontiers of mathematical science is quite remarkable.

The final years of Fourier's life were spent in Paris, where he was Secretary of the Académie des Sciences and succeeded Laplace as President of the Council of the Ecole Polytechnique. He died at the age of 62 on May 16, 1830.

PREFACE

This is an introductory treatment of Fourier series and their applications to boundary value problems in partial differential equations of engineering and physics. It is designed for students who have completed a first course in ordinary differential equations. In order that the book be accessible to as great a variety of readers as possible, there are footnotes to texts which give proofs of the more delicate results in advanced calculus that are occasionally needed. The physical applications, explained in some detail, are kept on a fairly elementary level.

The *first objective* of the book is to introduce the concept of orthonormal sets of functions and representations of arbitrary functions by series of functions from such sets. Representations of functions by Fourier series, involving sine and cosine functions, are given special attention. Fourier integral representations and expansions in series of Bessel functions and Legendre polynomials are also treated.

The *second objective* is a clear presentation of the classical method of separation of variables used in solving boundary value problems with the aid of those representations. In the final chapter, some attention is given to the verification of solutions and to their uniqueness, since the method cannot be presented properly without such considerations.

This book is a revision of its seventh edition, the first two of which were written by Professor Churchill alone. While improvements appearing in earlier revisions have been retained here, the entire book has been thoroughly rewritten. Some of the changes in this edition are mentioned below.

The regular Sturm-Liouville problems leading to Fourier cosine and sine series are treated by themselves in a separate section, and the same is true of the singular problems leading to Fourier cosine and sine integrals. It seemed that there were too many distractions when the solutions of those eigenvalue problems were obtained in the sections devoted mainly to illustrations of the method of separation of variables. A number of topics have been brought out of the problem sets and presented in their own sections, because of their special interest and importance. Examples of this are the Gibbs' phenomenon and the Poisson integral formula, together with the Sturm-Liouville problem involving periodic boundary conditions needed to obtain that formula. Another example is the derivation of a reduction formula to be used in evaluating integrals appearing in the coefficients of various Fourier-Bessel series.

Many other changes in this edition were suggested by readers who have spoken or written to me. Duhamel's principle, for instance, is discussed more thoroughly, and there are more physical problems using it later on. The chapter on Bessel functions now begins with a separate section on the gamma function in order to make the presentation of Bessel functions more efficient. Also, the Fourier-Bessel series found in this book are now listed in an appendix. While notation can vary from author to author, I have chosen to follow the classic text by Bartle that is listed in the Bibliography by changing to his notation for one-sided derivatives but keeping our notation in defining one-sided limits. Finally, it should be mentioned that problem sets appear even more frequently than in the last edition, in order to focus more directly on the material just introduced.

A *Student's Solutions Manual* (ISBN: 978-007-745415-9; MHID 007-745415-4) is available. It contains solutions to selected problems throughout the book.

This and earlier editions have benefited from the continued interest of friends, including current and former students. The late Ralph P. Boas, Jr., furnished the reference to Kronecker's extension of the method of integration by parts, and the derivation of the laplacian in cylindrical and spherical coordinates was suggested by a note of R. P. Agnew's in the *American Mathematical Monthly,* vol. 60, 1953. Finally, the most important source of support and encouragement was the staff at McGraw-Hill and my wife, Jacqueline Read Brown.

James Ward Brown

CONTENTS

CHAPTER 1

FOURIER SERIES

This book is concerned with two general topics:

(i) one is the representation of a given function by an infinite series involving a prescribed set of functions;
(ii) the other is a method of solving boundary value problems in partial differential equations, with emphasis on equations that are prominent in physics and engineering.

Representations by series are encountered in solving such boundary value problems. The theories of those representations can be presented independently. They have such attractive features as the extension of concepts of geometry, vector analysis, and algebra into the field of mathematical analysis. Their mathematical precision is also pleasing. But they gain in unity and interest when presented in connection with boundary value problems.

The set of functions that make up the terms in the series representation is determined by the boundary value problem. Representations by Fourier series, which are certain types of series of sine and cosine functions, are associated with a large and important class of boundary value problems. We shall give special attention to the theory and application of Fourier series and their generalizations. But we shall also consider various related representations, concentrating on those involving so-called Fourier integrals and what are known as Fourier-Bessel and Legendre series.

In this chapter, we begin our discussion of Fourier series. Once the convergence of such series has been established (Chap. 2) and a variety of partial differential equations have been derived (Chap. 3), we shall see (Chaps. 4 and 5) how such series are used in what is often referred to as the Fourier method for solving boundary value problems.

The first section here is devoted to a description of a class of functions that is central to the theory of Fourier series.

1. PIECEWISE CONTINUOUS FUNCTIONS

If the values $f(x)$ of a function f approach some finite number as x approaches x_0 from the right, the *right-hand limit* of f is said to exist at x_0 and is denoted by $f(x_0+)$. Thus

$$\lim_{\substack{x \to x_0 \\ x > x_0}} f(x) = f(x_0+).$$

The *left-hand limit* is similarly defined, so that

$$\lim_{\substack{x \to x_0 \\ x < x_0}} f(x) = f(x_0-).$$

EXAMPLE 1. Let the function f be defined for all nonzero x by means of the equations (see Fig. 1)

$$f(x) = \begin{cases} -x & \text{when } x < 0, \\ x + 1 & \text{when } x > 0. \end{cases}$$

Observe that the usual limit as x tends to zero does not exist. But

$$\lim_{\substack{x \to 0 \\ x > 0}} f(x) = f(0+) = 1$$

and

$$\lim_{\substack{x \to 0 \\ x < 0}} f(x) = f(0-) = 0.$$

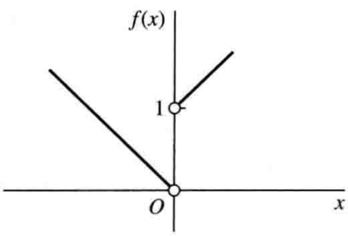


FIGURE 1

Let a function f be continuous at all points of a bounded open interval $a < x < b$ except possibly for a finite set of points $x_1, x_2, \ldots, x_{n-1}$, where

$$a < x_1 < x_2 < \cdots < x_{n-1} < b.$$

If we write $x_0 = a$ and $x_n = b$, then f is continuous on each of the n open subintervals

(1) $$x_0 < x < x_1, \quad x_1 < x < x_2, \quad \ldots, \quad x_{n-1} < x < x_n.$$