A FIRST COURSE

CALCULUS

RANSOM V. LYNCH
THE PHILLIPS EXETER ACADEMY

DONALD R. OSTBERG
NORTHERN ILLINOIS UNIVERSITY

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PREFACE

As asserted in the title, this is a text in calculus. It begins at the beginning, even a little before the beginning. It continues, with a fixity of purpose relieved only by a few digressions, to the point where most people agree a first course in the subject ends. It is neither encyclopedic nor barren of detail, neither revolutionary nor devoid of novelty. It was written with the student in mind, and we believe it is readable.

Although students beginning the study of calculus normally have had at least three years of high-school mathematics, it is the experience of the authors that this preparation is often inadequate. Indeed it has been said with some truth that more students fail to complete a first course in calculus because of an inability to manipulate algebraic expressions than for any other reason. We have therefore chosen to begin with a brief review of high-school algebra accompanied by a host of exercises, offering those students who need it an opportunity to sharpen their algebraic skills. Since this is definitely precalculus material it has been placed in the Preliminary Chapter; it counts not at all as far as calculus is concerned.

The text proper begins with Chapter 1, though here too the material is precalculus in that it does not involve any of the ideas that are usually associated with the word "calculus." Rather it introduces the idea of a function and some of the functions that are so important in calculus. Thus, although the student who has had a good grounding in algebra and in the theory of elementary functions can begin reading the text with Chapter 2, we recommend that he at least glance at Chapter 1 to make certain that the ideas it contains are familiar.

From Chapter 2 onward we have followed the dictates of tradition and logic in developing the subject. Admittedly, tradition is having a hard time of it these days, and sometimes logic as well. But in such a well-cultivated garden as calculus there is little to commend novelty for the sake of novelty, and much that argues against it. Thus we begin by introducing the derivative, though we do so gently because most students need time to get used to the ideas involved. We have also avoided excessive rigor at this point because, in our opinion, rigor at the outset of calculus obscures the beauty and power of the subject, confuses and alienates many competent students. After all, Newton managed to do reasonably well without ever producing a proof that would be acceptable in certain circles today.

In Chapter 4 we turn to the notion of the definite integral, which we approach traditionally through the problem of the area under a curve, citing Archimedes as our authority. Having introduced the notion of a limit in connection with the derivative, we are convinced that the student stands to gain an enhanced appreciation of the power of this idea by meeting it again in a completely different context. But again we have avoided excessive rigor in favor of a leisurely treatment of the ideas involved.

Thereafter organization and the order of events is as much a matter of choice as necessity, the whole subject now being open. We have chosen the order which looks first at logarithmic and exponential functions, returns to pick up the mean-value theorem and some of those almost obvious theorems relating the behavior of the derivative to the geometry of curves, and then continues with many of the familiar applications of differentiation and integration that we like to believe are part of our everyday world. One of the topics treated with particular care is differential equations, for there, in our opinion, is where almost all of the truly deep applications of calculus in the natural sciences are to be found. Moreover, we have tried, insofar as possible, to treat the subject of differential equations in a nonpedestrian fashion, focusing on ideas as much as on technique, on the types of problems that find their expression in differential equations as well as on the tricks that lead to their solution.

The text concludes with a chapter on infinite series, followed by a chapter on differential equations that ends by introducing the technique of undetermined coefficients to produce series solutions for second-order linear differential equations.

It is a long journey in one year, but one which we feel is both possible and rewarding.

Many people have made important contributions to the making of this book. In particular, we thank Henryetta Ponczek and Carole French, for typing, often above and beyond the call of duty; William Lipman, for working the exercises; Robert Herrera, Kenneth Skeen, and Donald Wilken, for many valuable suggestions that we have incorporated in the book, and no less for many others, possibly even more valuable, that we have chosen not to follow; George Springer, for perceptive advice and encouragement; and the people of Ginn-Blaisdell, for both skillful assistance and tactful forbearance. Any pedagogical virtues the book may have must be credited to those from whom we have learned whatever we know about teaching calculus: our teachers, colleagues, and students. Finally, we thank La Ru Lynch and Rosemarie Ostberg for their calm confidence that the job would be completed. Without that it never would have been.

Exeter, New Hampshire *R. V. L.*
DeKalb, Illinois *D. R. O.*

THE GREEK ALPHABET

A	α	alpha
B	β	beta
Γ	γ	gamma
Δ	δ	delta
E	ϵ	epsilon
Z	ζ	zeta
H	η	eta
Θ	θ ϑ	theta
I	ι	iota
K	κ	kappa
Λ	λ	lambda
M	μ	mu
N	ν	nu
Ξ	ξ	xi
O	o	omicron
Π	π ϖ	pi
P	ρ	rho
Σ	σ ς	sigma
T	τ	tau
Υ	υ	upsilon
Φ	ϕ φ	phi
X	χ	chi
Ψ	ψ	psi
Ω	ω	omega

CONTENTS

PRELIMINARY CHAPTER: A BRIEF REVIEW OF ALGEBRA

P.1 Introduction

Calculus, like plane geometry or any other branch of mathematics, is an abstract system consisting of certain assumptions, called axioms or postulates, and of some of the logical consequences of those assumptions, called theorems. The assumptions that form the basis for geometry are a set of statements about objects called points and lines, and those objects therefore form the subject matter of plane geometry. The assumptions at the foundations of calculus are a set of statements about objects called real numbers, and those therefore form the subject matter of calculus. The real numbers are familiar, for they also form the subject matter of arithmetic and of elementary algebra. Calculus, however, uses certain properties of real numbers largely ignored in more elementary work to develop the idea of a limit. This concept, which is the root idea of calculus, leads to techniques of tremendous power, applicable to a great variety of physical and mathematical problems.

It would be possible for us to trace the logical development of calculus from its basic assumptions, proving each theorem in rigorous detail, but this important and interesting task is better postponed to more advanced courses. The development which follows will therefore rely heavily on intuition and will proceed with a minimum of formal proof. This informality does not, however, imply that there will be no difficulties, or that the student can expect to achieve significant results without effort. Mathematics is not a spectator sport.

A. N. Whitehead, a mathematician and a philosopher, has said, "Without a doubt, technical facility is a first requisite for valuable mental activity: we shall fail to appreciate the rhythm of Milton, or the passion of Shelley, so long as we find it necessary to spell the words and are not quite sure of the forms of the individual letters."† The standard operations of elementary algebra occur frequently in applications of calculus, and unless the student can carry out these operations with reasonable facility he may lose the thread of an argument or be discouraged from pressing on to an interesting result. Therefore in this preliminary chapter we present a review of those topics in elementary algebra that seem to cause the most trouble in calculus.

P.2 Sets and intervals

SETS: Recognition and classification are among the most fundamental of mental activities. These particular activities involve a concept which is basic in mathematics, the concept of a set. Some familiar examples are the set of students in your mathematics course, the set of Sundays in the year 1900, the set of prime numbers, and the set of even primes.

We shall not attempt a formal definition of the word "set," nor of the concept of belonging to a set. Informally, we shall agree that a set has been adequately described when we have some way of deciding that a given object or element does or does not belong to the set. If the set contains only a finite number of elements, they can be listed, and we can determine whether an object does or does not belong to the set by comparing it with the list. Thus, for example, the set A of decimal digits can be described as

$$A = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}.$$

Note the use of a single letter, in this case A, as a name for the set, and also the use of braces, $\{\ \}$, to enclose the list of names of the elements of the set; these are conventions in general use. In the same way, we can designate the set of positive prime numbers less than 10 as

$$B = \{2, 3, 5, 7\}.$$

To indicate that an object belongs to a specified set we use the symbol $\in$, which is a modification of the Greek letter ϵ (epsilon). Thus, the notation "$x \in S$" should be read "x is an element of S," or "x belongs to S." With reference to the sets A and B just defined we may write $6 \in A$ (6 is an element of A), $3 \in B$ (3 belongs to B), and so on. To indicate that an object does not belong to a specified set, we draw a line through the $\in$ symbol: thus, $4 \notin$ B (4 does not belong to B). The reader should compare this notation with the familiar $\neq$ for "is not equal to" or "does not equal."

We say that a set P and a set Q are *equal*, $P = Q$, if P and Q contain exactly the same elements. In this connection we must be careful to distinguish between an object and a name for the object. A mathematics class is a set of people; the class roster is a list of their names. Our set B is a set of numbers; the symbols used in listing the elements of B are numerals, the names of numbers. A class is not changed if some members change their names, if the order of names is changed, or even if some of the names are repeated. Similarly,

$$B = \{\text{two}, \sqrt{9}, \tfrac{15}{3}, \text{VII}\} = \{2, 7, 5, 3\} = \{2, 2, 3, 5, 3, 7\}.$$

† *An Introduction to Mathematics* (Reprint) (New York: Oxford University Press, 1958), pp. 1–2.

But note that

$$B \neq \{2, 3, 5\} \qquad \text{and} \qquad B \neq \{2, 3, 5, 6, 7\}.$$

To list all the elements of a set is inconvenient if their number is large, impossible if it is infinite. In such cases the set can be specified by some description or rule. For example, the set C of all positive integers is specified by the description "the set of all positive integers." A standard and convenient way of writing this description is

$$C = \{x \colon x \text{ is an integer and } x > 0\},$$

read "C is the set of all x such that x is an integer and x is greater than 0." In general, the notation

$$\{\ldots \ : \ \ldots\},$$

called the *set-builder*, is read "the set of all . . . such that . . . ." Another example of the set-builder notation is

$$D = \{y \colon y \text{ is a real number and } -1 \leq y \leq 1\},$$

the set of all real numbers between -1 and 1, inclusive. We can also use this notation for the sets A and B already defined:

$$A = \{z \colon z \text{ is a decimal digit}\};$$
$$B = \{p \colon p \text{ is a prime number and } 0 < p < 10\}.$$

Given a set S, we may form new sets from selected elements of S; such sets are called subsets of S:

DEFINITION. *If each element of a set X is also an element of a set S, then X is a* subset *of S. This relation is denoted $X \subseteq S$.*

We also describe the relation $X \subseteq S$ by saying that X is included in S, or that S includes X. As a careful reading of the definition will show, it is entirely possible to have both $X \subseteq S$ and $S \subseteq X$; in this case $X = S$. It follows that for any set S, $S \subseteq S$. If $X \subseteq S$ but $X \neq S$, then there is at least one element $a \in S$ such that $a \in X$. In this case we say that X is a *proper subset* of S, or that S properly includes X, and write $X \subset S$. Note the analogy between the symbols "$\subset$" and "$\subseteq$" for set inclusion and the symbols "$<$" and "$\leq$" for the ordering of numbers. Indeed, if $n(X)$ denotes the number of elements in a finite set X, then $P \subset Q$ implies $n(P) < n(Q)$, and $P \subseteq Q$ implies $n(P) \leq n(Q)$.

It is important to distinguish between the $\in$ relation of set membership and the $\subset$ or $\subseteq$ relation of set inclusion. The statements $2 \in B$, $3 \in \mathrm{B}$, $\{2, 3\} \subset B$, and $\{2, 3\} \subseteq B$ are all true, but $2 \subset B$ and $\{2, 3\} \in B$ are false. The statement $\{2\} \subset B$ is also true; a set need not contain more than one element, and a set containing only one element is not the same object as the element itself, just as a bank account with only one dollar in it is not the same thing as one dollar.

The subsets of a given set can be combined in a variety of ways; there is even a substantial algebra of sets. The only operations of this algebra that we shall need to use, however, are union and intersection.

DEFINITION. *If P and Q are sets, then the* union *of P and Q, written $P \cup Q$, is the set of all elements that belong to at least one of P and Q:*

$$P \cup Q = \{x : x \in P \text{ or } x \in Q\}.$$

The word "or" in this definition is used in the standard mathematical sense: it means $x \in P$ or $x \in Q$ *or both.* As used in common speech, the word has two meanings. There is an exclusive "or," as in "Either I will go out or I will stay home"; it is not possible to do both. There is an inclusive "or," as in "Either I will study or I will watch television," a statement that will be true if I do both. It is essential in mathematics that this word have only one meaning, and mathematicians have therefore agreed to use it always in the inclusive sense.

The other set operation that we need is intersection.

DEFINITION. *If P and Q are sets, then the* intersection *of P and Q, written $P \cap Q$, is the set of all elements that belong to both P and Q:*

$$P \cap Q = \{x : x \in P \text{ and } x \in Q\}.$$

Example 1. Let

$$C = \{1, 2, 3\}, \qquad D = \{2, 4\}, \qquad E = \{3, 5\}.$$

Then

$$C \cup D = \{1, 2, 3, 4\}, \qquad C \cap D = \{2\},$$
$$D \cup E = \{2, 3, 4, 5\}, \qquad (C \cup D) \cup E = \{1, 2, 3, 4, 5\},$$
$$(C \cup E) \cap D = \{2\}.$$

Notice that the sets D and E in this example have no common elements. In this case $D \cup E$ causes no difficulty, $D \cup E = \{2, 3, 4, 5\}$, but does it make any sense to speak of $D \cap E$? It could be argued that D and E have no intersection. Yet it would be a great nuisance if, before we could talk about the intersection of two sets, we first had to check whether the sets actually had common elements. For this reason we introduce the concept of an *empty set*, $\emptyset$, that is, a set that does not contain any elements at all. This set can be defined in set-builder notation

$$\emptyset = \{x : x \neq x\}.$$

In our example, then, $D \cap E = \emptyset$. Two sets whose intersection is empty, like D and E in the example, are said to be *disjoint*.

It can be proved that there is only one empty set. Indeed, if $\emptyset$ is an empty set and G is also an empty set, then every element of $\emptyset$ (there aren't any) is also an element of G, and therefore $\emptyset \subseteq G$. Similarly, every element of G (there aren't any) is an element of $\emptyset$, and therefore $G \subseteq \emptyset$. Hence $G = \emptyset$. If this proof seems unconvincing, try turning it around: to prove that $\emptyset \neq G$, you must find an element in one of these sets that does not belong to the other. Since this is impossible, $\emptyset = G$. Furthermore, $\emptyset$ is a subset of every set: For given any set A, then each element of $\emptyset$ (there aren't any) is an element of A, and therefore $\emptyset \subseteq A$.

VARIABLES: The formulas of mathematics afford many examples of the use of a letter to represent any one of a set of numbers. For example, in the formula for the area of a circle, $A = \pi r^2$, it is generally understood that the radius r can be any positive number. In this case, r is used as a variable.

DEFINITION. *A* variable *is a symbol that stands for, and may be replaced by, any element of a specified set called the* domain *of the variable.*†

In calculus as in algebra, most of the variables used are letters of some alphabet, and most of them have sets of real numbers as their domains. If there is no danger of confusion, we shall therefore not bother, when describing domains, to keep repeating such specifications as "x is a real number." Thus, instead of writing

$$D = \{y: y \text{ is a real number and } -1 \leq y \leq 1\},$$

we shall simply write

$$D = \{y: -1 \leq y \leq 1\}.$$

It is important to note that D is a set of numbers. The variable y appears in our description of D, but $y \notin D$; it is a so-called "dummy" variable and could be replaced by any other symbol that does not already have a specific meaning attached to it. Thus,

$$\{x: -1 \leq x \leq 1\}, \qquad \{\Delta: -1 \leq \Delta \leq 1\},$$

and even

$$\{\text{John Doe}: -1 \leq \text{John Doe} \leq 1\}$$

are all names of the same set D.

INTERVALS: It is often illuminating to draw pictures representing sets or relations among sets. Calculus is based on certain properties of the set $\mathcal{R}$ of real numbers, and the standard picture illustrating this set is a number line. The idea of numbering the

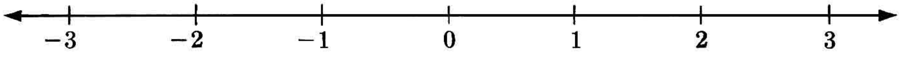

FIGURE P.1

points on a line is familiar to anyone who has used a ruler. The basic principle involved, which we assume to be valid, is that it is possible to construct a scale on any line so that there is a unique real number corresponding to each point of the line and a unique point corresponding to each real number. A line with such an associated number scale is called a *number line,* the number associated with each point is the *coordinate* of the point, and the set of all coordinates on a line is a *one-dimensional coordinate system.* The point with coordinate zero is the *origin* of the system. The correspondence thus established between points and numbers is such a natural one that we often find it convenient to use the word "point" to refer to a number.

† Strictly speaking, a variable is usually replaced by a symbol representing a specific element of its domain, and not by the element itself. We do not write numbers in place of variables in a formula, we write numerals, which are symbols for numbers. However, we shall not insist in this text on the distinction between things and the symbols that represent them.

Most of the variables encountered in calculus have as their domains certain subsets of the number line called intervals. If we think of the number line as a thin stick that can be broken into pieces, then the pieces, roughly speaking, are intervals. More precisely, an *interval* consists of either

(i) all points (numbers) on the number line that lie between two fixed points, called the *endpoints* of the interval, or

(ii) all points on the number line that lie to one side of a fixed point, called the *endpoint* of the interval.

In the first case the interval is said to be *finite* or *bounded;* in the second case it is said to be *infinite* or *unbounded.*† Thus, for instance, the set

$$D = \{x: -1 \leq x \leq 1\}$$

consisting of all real numbers between -1 and 1, inclusive, is a finite interval with endpoints -1 and 1. So is the set

$$I = \{x: -1 < x < 1\}.$$

Note, however, that D and I are not the same; D contains its endpoints while I does not. This difference, which at first sight may seem trifling, has important consequences in calculus, and for this reason intervals have been given names depending on whether or not they contain their endpoints. The following definition introduces these names and also the standard notation used to represent intervals.

DEFINITION. *Let a and b be real numbers, with $a < b$. Then*

(1) $\{x: a < x < b\}$ *is called the* open interval *from a to b, and is denoted* (a, b);

(2) $\{x: a \leq x \leq b\}$ *is called the* closed interval *from a to b, and is denoted* $[a, b]$;

(3) $\{x: b < x\}$ *is called the* open interval *from b to infinity, and is denoted* (b, ∞);

(4) $\{x: b \leq x\}$ *is called the* half-open interval *from b to infinity, and is denoted* $[b, \infty)$;

(5) $\{x: x < a\}$ *is called the* open interval *from minus infinity to a, and is denoted* $(-\infty, a)$;

(6) $\{x: x \leq a\}$ *is called the* half-open interval *from minus infinity to a, and is denoted* $(-\infty, a]$;

(7) $\{x: a \leq x < b\}$, *denoted* $[a, b)$, *and*

(8) $\{x: a < x \leq b\}$, *denoted* $(a, b]$, *are the* half-open intervals *from a to b; and*

(9) *the set of all real numbers is an* open interval, *denoted* $(-\infty, \infty)$.

Note the principle used in the notation introduced here: a square bracket is used when the corresponding endpoint belongs to the interval in question, and a parenthesis is used in all other cases. This notation has been followed in Figure P.2, where the

† The word "finite" refers to the length of the interval, not to the number of points it contains.