GARETH WILLIAMS

Mathematics with Applications in the Management, Natural, and Social Sciences

MATHEMATICS
with APPLICATIONS
in the MANAGEMENT,
NATURAL, and SOCIAL SCIENCES

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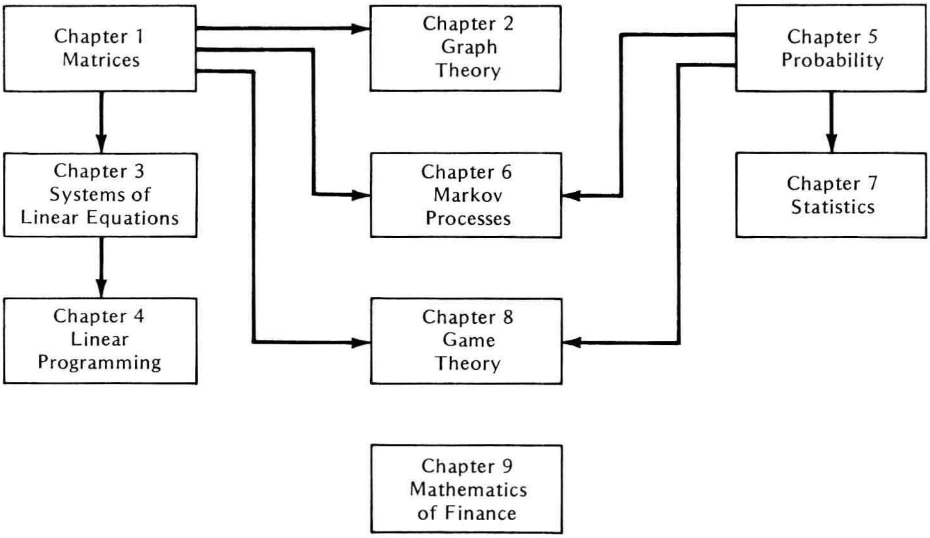
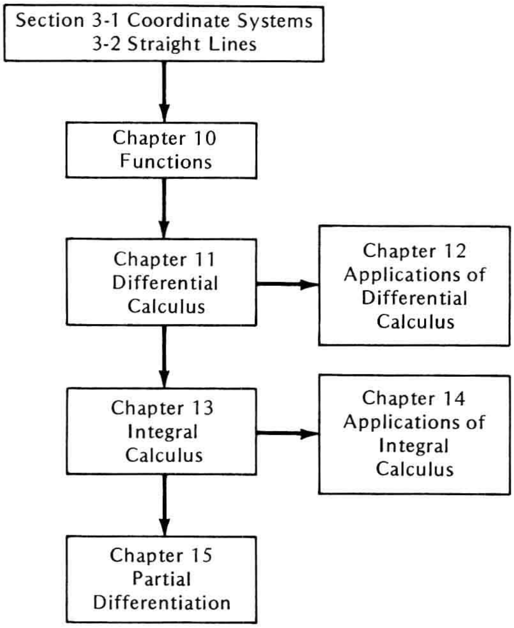
MATHEMATICS
with APPLICATIONS

Interdependence of Chapters

Finite Mathematics



Calculus



Chapter 16 Computing
Section 16-3 illustrates the
use of the computer in
1 Analyzing the motion of a projectile (Section 10-2)
2 Compound Interest (Section 9-2)
3 Analyzing population movements (Section 6-2)
4 Statistical Analysis (Section 7-1)

Preface

This text is designed for courses in Finite Mathematics and Calculus taken by students in Business, the Social and Behavioral Sciences. Four objectives have been kept in mind:

1. to introduce various areas of mathematics, such as linear algebra, statistics, and calculus, that are important to the users of mathematics;
2. to teach the reader when and how to use mathematical tools;
3. to convey an appreciation of the vital role played by mathematics in many areas of life;
4. to complement, in a mathematical manner, topics students meet in their own fields.

Theory is not stressed; however, a course such as this would not be complete without a certain amount of theory to provide a foundation. A wide variety of models and exercises allow the instructor to choose a desired level of rigor.

The art of using mathematics does not come naturally to many people. In this text the readers are introduced to models that have been carefully selected to develop the instinct for applying mathematics. They become involved in the models by carrying out extensions of given models. For example, in Section 6.2, a model for analyzing population movements is presented. The original model is criticized and refined and the reader carries out further modifications in the exercises. Instances of applications of the mathematics developed are given. For example, the Origin-Destination model that is used by highway departments uses matrix algebra. The City of Chicago has used this O-D model, and Denver is currently using it in planning a rapid transport system. Further examples include discussions of interdependency of industries in the United States, land use in center-city Toronto, and preference patterns for comic strips among teenagers. I believe that technical examples of applications and research results can often be suitably summarized and explained to readers at this level. The mathematics is thus developed within an atmosphere of real application. The models introduced cover a variety of fields and vary in depth of coverage and degree of difficulty. I have not restricted the models to the areas of Business, the Social and Behavioral Sciences; they cover a broad spectrum of fields. A model has occasionally been selected because it illustrates well an important mathematical technique or concept. Furthermore, models developed in one area often become useful either in direct form or in modified form in other fields. I have attempted to include this flavor of modeling. There are more models than can be covered in a single course; thus, the instructor can select those most suitable for the class. All discussions are self-contained since audiences in this course are often of varied backgrounds.

An important aim of this course, I believe, is to complement in a mathematical manner, topics students meet in their own disciplines – to give a mathematical orientation to those topics without duplicating the material covered in other courses. Business students for example, are instructed on the importance of marginal analysis in Business courses. However, the topic is not usually presented from a mathematical viewpoint; the role of calculus in the theory is not mentioned. Here I have attempted to give the mathematical foundation of marginal analysis that will round off the knowledge that students obtain in the Business School.

I have included reviews of properties of real numbers and of basic algebra in the appendixes rather than at the beginning of the book. The review of the real numbers can take place, if desired, at the commencement of the Finite Mathematics part of the course. The student will need to be able to manipulate exponents in probability theory for example. However, the algebra review (if necessary) should, I believe, take place just before Chapter 10, the chapter on functions. Finite Mathematics does not equal precalculus mathematics. The algebra review paves the way for calculus.

The chapter on functions has been located immediately before the chapters on calculus. This I feel is its natural location. Objective functions of linear programming and probability functions have been introduced earlier. These concepts can be understood without delving into the function aspect. With this arrangement of topics the text divides naturally into a Finite Mathematics part, Chapters 1–9, and a Calculus part, 10–16.

Most of the material in this text was developed in courses taught at Stetson University and the University of Denver. It has been used in the mathematics course for social and behavioral sciences, business and economics, and also in the liberal arts course that many students take to satisfy a science requirement. In addition, the material has been used in an undergraduate models course. A number of the models included were developed in conjunction with students at these institutions.

Applying mathematics is fun – I hope that some of this spirit will rub off on the reader.

I express my appreciation to the many colleagues with whom I have corresponded over the writing of this text. I thank colleagues Gene W. Medlin, Josephine Cannon, Dennis J. Kletzing, and Bruce Bradford of Stetson University. My deepest gratitude goes to Raymond J. Cannon of Baylor University for his proofreading and for the many valuable discussions we have had concerning many aspects of this course.

I would also like to express my appreciation to the reviewers of the text for their constructive comments: Ruth Hoffman, University of Denver; Edward Spitznagel, Jr., Washington University; Jean Bevis, Georgia State University; Clay Ross, Jr., The University of the South; Marialuisa McAllister, Moravian College; Manfred Stoll, University of South Carolina; Charles Cheney, Indiana State University; John S. Klein, Hobart and William Smith Colleges; Paul B. Burcham, University of Missouri-Columbia; Richard Ringiesen, Clemson University; Maurice Monahan, South Dakota State University; Hal J. Forsey, San Francisco University; Mark A. Cole, University of Vermont; Richard E. Goodrick, California State University-Hayward; Alex Kleiner, Drake University; Norman Locksley, Prince George's Community College; Laurence Small, Los Angeles Pierce College; Frank Sheehan, San Francisco State College; and Raymond Cannon, Baylor University.

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I am most grateful to Kathy McCormick for her typing. My special thanks go to my wife Donna for her typing contribution, her valuable suggestions concerning the contents of the text, and her encouragement.

G. W.

Contents

PART 1. FINITE MATHEMATICS

CHAPTER 5 PROBABILITY

CHAPTER 6 MARKOV PROCESSES

CHAPTER 7 STATISTICS

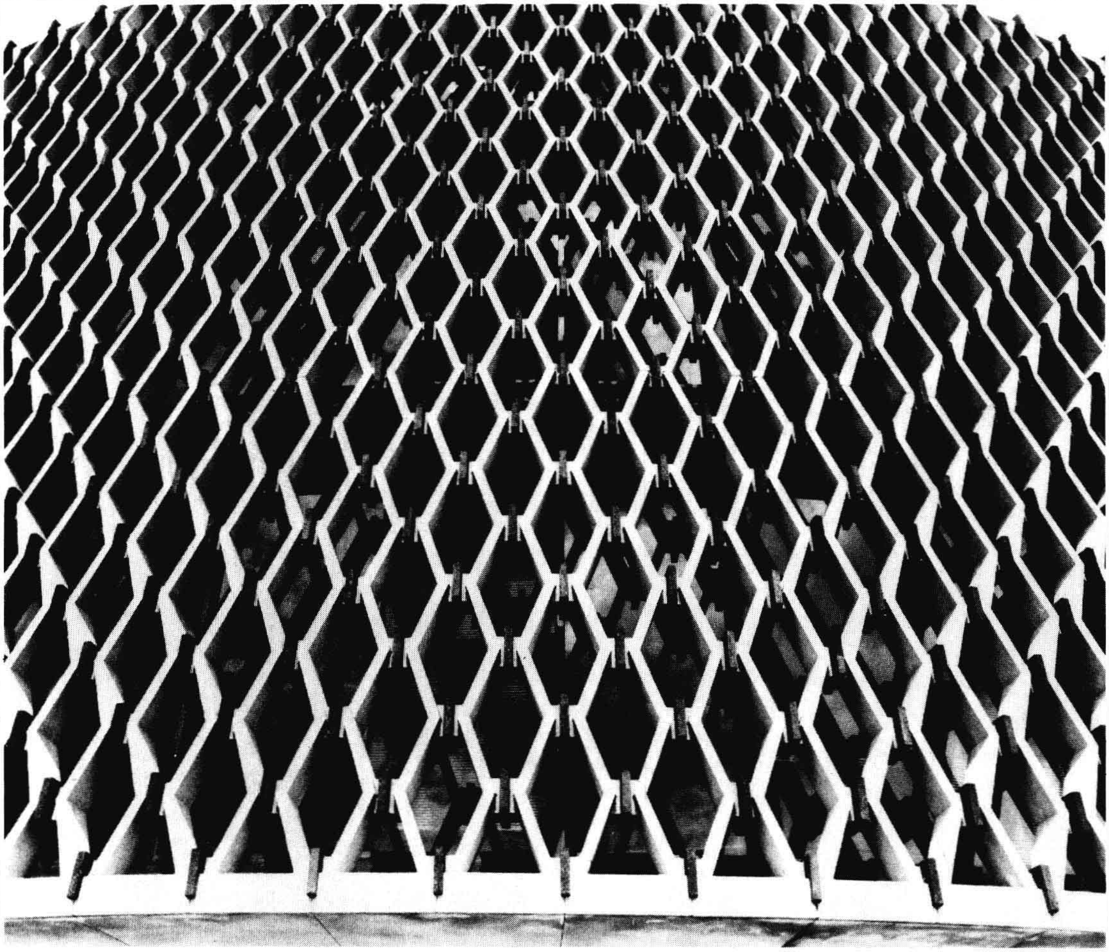
CHAPTER 8 GAME THEORY

CHAPTER 9 MATHEMATICS OF FINANCE

PART 2. CALCULUS

CHAPTER 10 FUNCTIONS

APPENDIXES



MATRICES

1

1–1 INTRODUCTION TO MATRICES

The aim of this course is to provide the readers with an understanding of certain areas of modern mathematics and to verse them in the skills required to use mathematics. Mathematics exists as an art in its own right. It is, however, more than that – it is also the language of modern science and business. We shall develop mathematical concepts to give the reader an appreciation of the elegance of the subject, and we shall introduce applications to give the reader insight into the roles mathematics plays in today's world.

In this chapter we develop the theory of matrices; in later chapters we shall see the many uses of these tools. Consider the set of real numbers: 1, 0.5, -3.6, and so on. We are all very familiar with this set and find the elements to be very useful in our daily lives. In fact, imagine a society without the real numbers! These numbers were developed and their properties investigated by mathematicians. Surprisingly, it was not until the late nineteenth century that a systematic analysis of the real number system was carried out.

Our aim will not be to delve deeply into the properties of the real numbers, for this becomes a highly specialized area of mathematics. Our goal will be to extend some of the ideas involved. This will be an interesting task mathematically, and the theory we develop – the theory of matrices – has widespread applications. This mathematical language is used in such diverse fields as psychology, biology, physics, and business.

A natural generalization of real numbers is to consider pairs of real numbers, such as $(1 \quad 2)$, $(3 \quad -1)$, and $(.25 \quad .75)$, in place of single numbers. But why limit the concept to pairs? Let us consider triples, such as $(1 \quad -3 \quad 3)$, $(2 \quad 1 \quad 0)$, and so on, and larger arrays, such as $(1 \quad 5 \quad 2 \quad 6 \quad 3 \quad -4)$. Let us also work vertically and construct rectangular arrays, such as $\begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 3 \\ 3 & 2 \\ 1 & 2 \end{pmatrix}$. At this stage, we are justified in feeling that we have made a breakthrough in our thinking. We are no longer restricted to the world of real numbers; we are aware of an extension of the concept. Before proceeding further, let us consolidate our position by giving such a rectangular array a name: *matrix.*

DEFINITION 1: A *matrix* is a rectangular array of numbers. Some examples of matrices are

$$\begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 \\ 2 \\ 3 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 3 & 4 \\ 1 & 2 & 3 \end{pmatrix} \qquad ■$$

Matrices arise very naturally in numerous situations. The reader will see their use throughout this text.

Example 1 The following type of matrix is used to convey statistics (the number of games won, lost, or tied) in the National Football League. This particular matrix represents positions in the Western Division.

Example 1
(continued)

$$\begin{array}{lc} & \begin{array}{ccc} \text{W} & \text{L} & \text{T} \end{array} \\ \begin{array}{l}\text{Oakland}\\ \text{Denver}\\ \text{Kansas City}\\ \text{San Diego}\end{array} & \begin{pmatrix} 4 & 1 & 0\\ 2 & 2 & 1\\ 2 & 3 & 0\\ 1 & 4 & 0\end{pmatrix}\end{array}$$

■

The type of matrix discussed in the following example is used in the *Leontief input-output model* in economics. This model is used to analyze interdependence of industries.

Example 2

Consider an economic situation involving three interdependent industries, with each industry producing a single commodity. The output of any one industry is needed as input by the other industries, and also by the industry itself. This interdependence can be described by a matrix such as

$$\begin{array}{cc} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c}1\\ 2\\ 3\end{array} & \begin{pmatrix} .25 & .40 & .50\\ .35 & .10 & \textcircled{.20}\\ .20 & .30 & .10\end{pmatrix}\end{array}$$

We have labeled the industries as 1, 2, and 3. The interpretation of the elements of the matrix is as follows:

Consider the 0.20 in row 2, column 3 of this matrix (circled). The implication is that 20¢ worth of commodity 2 is required to produce $1 of commodity 3. The 0.40 in row 1, column 2 implies that 40¢ worth of commodity 1 is required to produce $1 of commodity 2, and so on.

We shall discuss the Leontief input-output model further in Section 3–8. ■

Example 3

Highway divisions use origin-destination data in mathematical models to analyze traffic patterns. These data are based on door-to-door questionnaires and are drawn up in the form of an origin-destination matrix (O-D matrix). The region of interest is divided into zones, and an estimate of the daily traffic between zones is made. The following matrix could represent daily traffic in a small city that has been divided into four zones, I through IV:

$$\begin{array}{ccc} & & \text{Destination zone} \\ & & \begin{array}{cccc} \text{I} & \text{II} & \text{III} & \text{IV} \end{array} \\ \begin{array}{c}\\ \text{Origin}\\ \text{zone}\\ \\ \end{array} & \begin{array}{c}\text{I}\\ \text{II}\\ \text{III}\\ \text{IV}\end{array} & \begin{pmatrix} 2{,}000 & 1{,}500 & 200 & 700\\ 1{,}400 & 3{,}000 & 900 & 360\\ 175 & 800 & 2{,}500 & 1{,}700\\ 600 & 400 & 2{,}100 & 1{,}000\end{pmatrix}\end{array}$$

To interpret the matrix, look up the row of the origin zone of interest and the column of the destination zone. The number at the intersection of the row and column represents the daily traffic flow in automobiles. For example, the daily traffic flow from zone II to zone IV is 360 automobiles.

Highway divisions use O-D matrices to analyze existing traffic patterns and to predict future highway needs. The city of Chicago used a mathematical model based on such an O-D matrix: the *gravity model.* (The model takes its name from Newton's law of gravity, from which it was developed.) Ideally, the highway division works with the local planning authority in applying such a model. Currently, a regional transit system is being planned for Denver and the surrounding region. It is to be completed by 1983 and will include a computerized rapid transit network, a regional bus service, and a local bus service. The planners for this system are

Example 3 (continued) working jointly with the Denver Regional Council of Governments and the Colorado Division of Highways. The gravity model is being used in deciding the various routes to be operated by the transit system.[1] ■

We have seen examples of matrices of various shapes and sizes. We can classify a matrix by the number of rows and columns that it has. For example, $\begin{pmatrix} 1 & 2 & 5 \\ 3 & 4 & 6 \end{pmatrix}$ is a 2 × 3 matrix, for it has two rows and three columns. $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & 7 & 6 \\ 4 & 4 & 3 & 2 \end{pmatrix}$ is a 3 × 4 matrix, having three rows and four columns. The convention used in describing the size of a matrix is to state the number of rows first, followed by the number of columns.

A matrix with the same number of rows as columns is said to be a *square matrix.* Two matrices are of the *same kind* if they have the same number of rows and the same number of columns. For example,

$$\begin{pmatrix} 1 & 3 & 2 \\ -1 & 3 & 4 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 2 & -1 & 7 \\ 6 & 3 & 4 \end{pmatrix}$$

are matrices of the same kind; they are both 2 × 3 matrices. Two matrices of the same kind are said to be *equal* if and only if their corresponding components are equal.

Let us now attempt to develop a theory of matrices that is motivated by the structure that we have for the set of real numbers (rules of arithmetic). Addition is defined for the set of real numbers; it has proven to be very useful. Now we will define addition of matrices. It would seem natural to add $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $\begin{pmatrix} 1 & 3 \\ -1 & 2 \end{pmatrix}$ as follows:

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 1 & 3 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 1+1 & 2+3 \\ 3-1 & 4+2 \end{pmatrix} = \begin{pmatrix} 2 & 5 \\ 2 & 6 \end{pmatrix}$$

Here we have added corresponding elements. It is apparent that we cannot add matrices that are not of the same kind in this manner, for their elements do not correspond. Guided by this discussion, we arrive at the following definition.

> DEFINITION 2: The sum of two matrices of the same kind is obtained by adding corresponding elements. If two matrices are not of the same kind, they cannot be added; we say that their sum does not exist. Subtraction is performed on matrices of the same kind by subtracting corresponding elements. ■

Example 4 Consider the following matrices:

$$A = \begin{pmatrix} -1 & 2 & 3 \\ 0 & 1 & 4 \end{pmatrix} \quad B = \begin{pmatrix} 3 & -2 & 6 \\ 7 & -1 & 2 \end{pmatrix} \quad C = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}$$

---

[1] For further discussion of various origin-destination models, see Anthony J. Catanese, ed., *Perspectives in Urban Transportation Research* (Lexington Books, 1972).

Example 4 (continued)

Determine the sums $A + B$ and $A + C$ if possible.

$$A + B = \begin{pmatrix} -1 & 2 & 3 \\ 0 & 1 & 4 \end{pmatrix} + \begin{pmatrix} 3 & -2 & 6 \\ 7 & -1 & 2 \end{pmatrix}$$
$$= \begin{pmatrix} -1+3 & 2-2 & 3+6 \\ 0+7 & 1-1 & 4+2 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 9 \\ 7 & 0 & 6 \end{pmatrix}$$

The sum $A + C$ does not exist since A and C are not matrices of the same kind. (Try adding these matrices using the rule.) ■

Let us extend our definition to enable us to add more than just two matrices. For example, let us define the following:

$$\begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 3 & 4 \\ 2 & 1 \end{pmatrix} + \begin{pmatrix} 5 & 2 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1+3+5 & 2+4+2 \\ 0+2-1 & -1+1+0 \end{pmatrix} = \begin{pmatrix} 9 & 8 \\ 1 & 0 \end{pmatrix}$$

We add a string of matrices that are of the same kind in this natural manner by adding corresponding elements. This rule for adding matrices is useful, as the following example illustrates.

Example 5

A clinic has three doctors, each with his own speciality. Patients attending the clinic see more than one doctor. The accounts are drawn up monthly and handled systematically through the use of matrices. We illustrate this accounting with four patients. This illustration can easily be extended to accommodate any number of patients.

$$\begin{array}{ccc} & & \text{Doctor} \\ & & \begin{matrix} \text{I} & \text{II} & \text{III} \end{matrix} \\ \text{Patient} & \begin{matrix} \text{A} \\ \text{B} \\ \text{C} \\ \text{D} \end{matrix} & \begin{pmatrix} 10 & 25 & 0 \\ 0 & 20 & 40 \\ 15 & 0 & 25 \\ 10 & 10 & 0 \end{pmatrix} \end{array}$$

The entries in the above matrix are in dollars and represent the bill from each doctor to each patient for a certain month.

During a given quarter there would be three such matrices. To determine the quarterly bill to each patient from each doctor, add the three matrices. Assume that the monthly bills are as follows:

$$\begin{array}{cccccccc} & \text{First month} & & \text{Second month} & & \text{Third month} & & \text{Quarter} \\ & \begin{matrix} \text{I} & \text{II} & \text{III} \end{matrix} & & \begin{matrix} \text{I} & \text{II} & \text{III} \end{matrix} & & \begin{matrix} \text{I} & \text{II} & \text{III} \end{matrix} & & \begin{matrix} \text{I} & \text{II} & \text{III} \end{matrix} \\ \begin{matrix} \text{A} \\ \text{B} \\ \text{C} \\ \text{D} \end{matrix} & \begin{pmatrix} 10 & 25 & 0 \\ 0 & 20 & 40 \\ 15 & 0 & 25 \\ 10 & 10 & 0 \end{pmatrix} & + & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 100 & 0 \\ 10 & 0 & 0 \\ 20 & 0 & 0 \end{pmatrix} & + & \begin{pmatrix} 20 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 20 \\ 15 & 15 & 0 \end{pmatrix} & = & \begin{pmatrix} 30 & 25 & 0 \\ 0 & 120 & 40 \\ 25 & 0 & 45 \\ 45 & 25 & 0 \end{pmatrix} \end{array}$$

Thus, A's bill from doctor I during the quarter would total \$30; B's quarterly bill from doctor III would total \$40; and so on.

Although this analysis and others like it in this text can theoretically be carried out without the use of matrices, the handling of large quantities of data is often most efficiently done on computers using matrix techniques and storage facilities. ■