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Second Edition

T. A. Springer

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Second Edition

T.A. Springer

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Modern Birkhäuser Classics

Many of the original research and survey monographs in pure and applied mathematics published by Birkhäuser in recent decades have been groundbreaking and have come to be regarded as foundational to the subject. Through the MBC Series, a select number of these modern classics, entirely uncorrected, are being re-released in paperback (and as eBooks) to ensure that these treasures remain accessible to new generations of students, scholars, and researchers.

Preface to the Second Edition

This volume is a completely new version of the book under the same title, which appeared in 1981 as Volume 9 in the series "Progress in Mathematics," and which has been out of print for some time. That book had its origin in notes (taken by Hassan Azad) from a course on the theory of linear algebraic groups, given at the University of Notre Dame in the fall of 1978. The aim of the book was to present the theory of linear algebraic groups over an algebraically closed field, including the basic results on reductive groups. A distinguishing feature was a self-contained treatment of the prerequisites from algebraic geometry and commutative algebra.

The present book has a wider scope. Its aim is to treat the theory of linear algebraic groups over arbitrary fields, which are not necessarily algebraically closed. Again, I have tried to keep the treatment of prerequisites self-contained.

While the material of the first ten chapters covers the contents of the old book, the arrangement is somewhat different and there are additions, such as the basic facts about algebraic varieties and algebraic groups over a ground field, as well as an elementary treatment of Tannaka's theorem in Chapter 2. Errors – mathematical and typographical – have been corrected, without (hopefully) the introduction of new errors. These chapters can serve as a text for an introductory course on linear algebraic groups.

The last seven chapters are new. They deal with algebraic groups over arbitrary fields. Some of the material has not been dealt with before in other texts, such as Rosenlicht's results about solvable groups in Chapter 14, the theorem of Borel of Tits on the conjugacy over the ground field of maximal split torus in an arbitrary linear algebraic group in Chapter 15 and the Tits classification of simple groups over a ground field in Chapter 17.
The prerequisites from algebraic geometry are dealt with in Chapter 11.

I am grateful to many people for comments and assistance: P. Hewitt and Zhe-Xian Wang sent me several years ago lists of corrections of the second printing of the old book, which were useful in preparing the new version. A. Broer, Konstanze Rietsch and W. Soergel communicated lists of comments on the first part of the present book and K. Bongartz, J. C. Jantzen. F. Knop and W. van der Kallen commented on points of detail. The latter also provided me with pictures, and W. Casselman provided Dynkin and Tits diagrams. A de Meijer gave frequent help in coping with the mysteries of computers.

Lastly. I thank Birkhäuser – personified by Ann Kostant– for the help (and patience) with the preparation of this second edition.

T. A. Springer

Linear Algebraic Groups

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Chapter 1

Some Algebraic Geometry

This preparatory chapter discusses basic results from algebraic geometry, needed to deal with the elementary theory of algebraic groups. More algebraic geometry will appear as we go along. More delicate results involving ground fields are deferred to Chapter 11.

1.1. The Zariski topology

1.1.1. Let k be an algebraically closed field and put $V = k^n$. The elements of the polynomial algebra $S = k[T_1, \dots, T_n]$ (abbreviated to $k[T]$) can be viewed as k-valued functions on V. We say that $v \in V$ is a *zero* of $f \in k[T]$ if $f(v) = 0$ and that v is a zero of an ideal I of S if $f(v) = 0$ for all $f \in I$. We denote by $\mathcal{V}(I)$ the set of zeros of the ideal I. If X is any subset of V, let $\mathcal{I}(X) \subset S$ be the ideal of the $f \in S$ with $f(v) = 0$ for all $v \in X$.

Recall that the *radical* or nilradical $\sqrt{I}$ of the ideal I (see [Jac5, p. 392]) is the ideal of the $f \in S$ with $f^n \in I$ for some integer $n \geq 1$. A *radical ideal* is one that coincides with its radical. It is obvious that all $\mathcal{I}(X)$ are radical ideals.

We shall need Hilbert's Nullstellensatz in two equivalent formulations.

1.1.2. Proposition. *(i) If I is a proper ideal in S then $\mathcal{V}(I) \neq \emptyset$;*
(ii) For any ideal I of S we have $\mathcal{I}(\mathcal{V}(I)) = \sqrt{I}$.

For a proof see for example [La2, Ch. X, §2] . The proposition can also be deduced from the results of 1.9 (see Exercise 1.9.6 (2)).

1.1.3. Zariski topology on V. The function $I \mapsto \mathcal{V}(I)$ on ideals has the following properties:
(a) $\mathcal{V}(\{0\}) = V$, $\mathcal{V}(S) = \emptyset$;
(b) *If $I \subset J$ then $\mathcal{V}(J) \subset \mathcal{V}(I)$;*
(c) $\mathcal{V}(I \cap J) = \mathcal{V}(I) \cup \mathcal{V}(J)$;
(d) *If $(I_\alpha)_{\alpha \in A}$ is a family of ideals and $I = \sum_{\alpha \in A} I_\alpha$ is their sum, then $\mathcal{V}(I) = \bigcap_{\alpha \in A} \mathcal{V}(I_\alpha)$.*

The proof of these properties is left to the reader (*Hint*: for (c) use that $I.J \subset I \cap J$). It follows from (a), (c) and (d) that there is a topology on V whose closed subsets are the $\mathcal{V}(I)$, I running through the ideals of S. This is the *Zariski topology*. The induced topology on a subset X of V is the Zariski topology of V. A closed set in V is called an *algebraic set*.

1.1.4. Exercises. (1) Let $V = k$. The proper algebraic sets are the finite ones.

(2) The Zariski closure of $X \subset V$ is $\mathcal{V}(\mathcal{I}(X))$.
(3) The map $\mathcal{I}$ defines an order reversing bijection of the family of Zariski closed subsets of V onto the family of radical ideals of S. Its inverse is $\mathcal{V}$.
(4) The Euclidean topology on $\mathbf{C}^n$ is finer than the Zariski topology.

1.1.5. Proposition. *Let $X \subset V$ be an algebraic set.*
(i) The Zariski topology of X is T_1, i.e., points are closed;
(ii) Any family of closed subsets of X contains a minimal one;
(iii) If $X_1 \supset X_2 \supset \dots$ is a descending sequence of closed subsets of X, there is an h such that $X_i = X_h$ for $i \geq h$;
(iv) Any open covering of X has a finite subcovering.

If $x = (x_1, \dots, x_n) \in X$ then x is the unique zero of the ideal of S generated by $T_1 - x_1, \dots, T_n - x_n$. This implies (i). (ii) and (iii) follow from the fact that S is a Noetherian ring [La2, Ch. VI, §1], using 1.1.4 (3).

To establish assertion (iv) we formulate it in terms of closed sets. We then have to show: if $(I_\alpha)_{\alpha \in A}$ is a family of ideals such that $\bigcap_{\alpha \in A} \mathcal{V}(I_\alpha) = \emptyset$, there is a finite subset B of A such that $\bigcap_{\alpha \in B} \mathcal{V}(I_\alpha) = \emptyset$. Using properties (a), (d) of 1.1.3 and 1.1.4 (3) we see that $\sum_{\alpha \in A} I_\alpha = S$. There are finitely many of the I_α, say $I_1, \dots, I_h$, such that 1 lies in their sum. It follows that $I_1 + \dots + I_h = S$, which implies that $\bigcap_{i=1}^{h} \mathcal{V}(I_i) = \emptyset$. □

A topological space X with the property (ii) is called *noetherian*. Notice that (ii) and (iii) are equivalent properties (compare the corresponding properties in noetherian rings, cf. [La2, p. 142]. X is *quasi-compact* if it has the property of (iv).

1.1.6. Exercise. A closed subset of a noetherian space is noetherian for the induced topology.

1.2. Irreducibility of topological spaces

1.2.1. A topological space X (assumed to be non-empty) is *reducible* if it is the union of two proper closed subsets. Otherwise X is *irreducible*. A subset $A \subset X$ is irreducible if it is irreducible for the induced topology. Notice that X is irreducible if and only if any two non-empty open subsets of X have a non-empty intersection.

1.2.2. Exercise. An irreducible Hausdorff space is reduced to a point.

1.2.3. Lemma. *Let X be a topological space.*
(i) $A \subset X$ is irreducible if and only if its closure $\bar{A}$ is irreducible;
(ii) Let $f : X \to Y$ be a continuous map to a topological space Y. If X is irreducible then so is the image fX.

Let A be irreducible. If $\bar{A}$ is the union of two closed subsets A_1 and A_2 then A is the union of the closed subsets $A \cap A_1$ and $A \cap A_2$. Because of the irreducibility of A, we have (say) $A \cap A_1 = A$, and $A \subset A_1, \bar{A} \subset A_1$. So $\bar{A}$ is irreducible.

Conversely, assume this to be the case. If A is the union of two closed subsets $A \cap B_1, A \cap B_2$, where B_1, B_2 are closed in X, then $\bar{A} \subset B_1 \cup B_2$. It follows that $\bar{A} \cap B_1 = \bar{A}$, whence $A \cap B_1 = A$. The irreducibility of A follows.

The proof of (ii) is easy and can be omitted. □

1.2.4. Proposition. *Let X be a noetherian topological space. Then X has finitely many maximal irreducible subsets. These are closed and cover X.*

It is clear from 1.2.3 (i) that maximal irreducible subsets of X are closed.

Next we claim that X is a union of finitely many irreducible closed subsets. Assume this to be false. Then the noetherian property 1.1.5 (ii) and 1.1.6 imply that there is a minimal non-empty closed subset A of X which is not a finite union of irreducible closed subsets. But A must be reducible, so it is a union of two proper closed subsets. Because of the minimality of A these have the property in question, and a contradiction emerges. This establishes the claim.

Let $X = X_1 \cup ... \cup X_s$, where the X_i are irreducible and closed. We may assume that there are no inclusions among them. If Y is an irreducible subset of X then $Y = (Y \cap X_1) \cup ... \cup (Y \cap X_s)$ and by the definition of irreducibility we must have $Y \subset X_i$ for some i, i.e., any irreducible subset of X is contained in one of the X_i. This implies that the X_i are the maximal irreducible subsets of X. The proposition follows. □

The maximal irreducible subsets of X are called the (irreducible) *components* of X.

We now return to the Zariski topology on $V = k^n$.

1.2.5. Proposition. *A closed subset X of V is irreducible if and only if $\mathcal{I}(X)$ is a prime ideal.*

Let X be irreducible and let $f, g \in S$ be such that $fg \in \mathcal{I}(X)$. Then

$$X = (X \cap \mathcal{V}(fS)) \cup (X \cap \mathcal{V}(gS))$$

and the irreducibility of X implies that (say) $X \subset \mathcal{V}(fS)$, which means that $f \in \mathcal{I}(X)$. It follows that $\mathcal{I}(X)$ is a prime ideal.

Conversely, assume this to be the case and let $X = \mathcal{V}(I_1) \cup \mathcal{V}(I_2) = \mathcal{V}(I_1 \cap I_2)$. If $X \neq \mathcal{V}(I_1)$, then there is $f \in I_1$ with $f \notin \mathcal{I}(X)$. Since $fg \in \mathcal{I}(X)$ for all $g \in I_2$ it follows from the primeness of $\mathcal{I}(X)$ that $I_2 \subset \mathcal{I}(X)$, whence $X = \mathcal{V}(I_2)$. So X is irreducible. □

1.2.6. Exercises. (1) Let X be a noetherian space. The components of X are its

maximal irreducible closed subsets.
(2) Any radical ideal I of S is an intersection $I = P_1 \cap ... \cap P_s$ of prime ideals. If there are no inclusions among them, they are uniquely determined, up to order.

1.2.7. Recall that a topological space is *connected* if it is not the union of two disjoint proper closed subsets. An irreducible space is connected. The following exercises give some results on connectedness and the relation with the notion of irreducibility.

1.2.8. Exercises. (1) (a) A noetherian space X is a disjoint union of finitely many connected closed subsets, its connected components. They are uniquely determined.
(b) A connected component of X is a union of irreducible components.
(2) A closed subset X of $V = k^n$ is not connected if and only if there are two ideals I_1, I_2 in S with $I_1 + I_2 = S$, $I_1 \cap I_2 = \mathcal{I}(X)$.
(3) Let $X = \{(x, y) \in k^2 \mid xy = 0\}$. Then X is a closed subset of k^2 which is connected but not irreducible.

1.3. Affine algebras

1.3.1. We now turn to more intrinsic descriptions of algebraic sets. Let $X \subset V$ be one. The restriction to X of the polynomial functions of S form a k-algebra isomorphic to $S/\mathcal{I}(X)$, which we denote by $k[X]$. This algebra has the following properties:
(a) $k[X]$ *is a k-algebra of finite type*, i.e., there is a finite subset $\{f_1, \dots, f_r\}$ of $k[X]$ such that $k[X] = k[f_1, \dots, f_r]$;
(b) $k[X]$ *is reduced*, i.e., 0 is the only nilpotent element of $k[X]$.

A k-algebra with these two properties is called an *affine k-algebra*. If A is an affine k-algebra, then there is an algebraic subset X of some k^r such that $A \simeq k[X]$. For $A \simeq k[T_1, \dots, T_r]/I$, where I is the kernel of the homomorphism sending the T_i to the generator f_i of A (as in (a)), then A is reduced if and only if I is a radical ideal. We call $k[X]$ the *affine algebra* of X.

1.3.2. We next show that the algebraic set X and its Zariski topology are determined by the algebra $k[X]$.

If I is an ideal in $k[X]$ let $\mathcal{V}_X(I)$ be the set of the $x \in X$ with $f(x) = 0$ for all $f \in I$. If Y is a subset of X let $\mathcal{I}_X(Y)$ be the ideal in $k[X]$ of the f such that $f(y) = 0$ for all $y \in Y$. If A is any affine algebra, let $\mathrm{Max}(A)$ be the set of its maximal ideals. If X is as before and $x \in X$, then $M_x = \mathcal{I}_X(\{x\})$ is a maximal ideal (because $k[X]/M_x$ is isomorphic to the field k).

1.3.3. Proposition. *(i) The map* $x \mapsto M_x$ *is a bijection of* X *onto* $\mathrm{Max}(k[X])$, *moreover* $x \in \mathcal{V}_X(I)$ *if and only if* $I \subset M_x$*;*
(ii) The closed sets of X *are the* $\mathcal{V}_X(I)$, *I running through the ideals of* $k[X]$.

Since $k[X] \simeq S/\mathcal{I}(X)$ the maximal ideals $k[X]$ correspond to the maximal ideals of S containing $\mathcal{I}(X)$. Let M be a maximal ideal of S. Then 1.1.4 (3) and 1.1.5 (ii) imply that M is the set of all $f \in S$ vanishing at some point of k^n. From this the first point of (i) follows, and the second point is obvious. (ii) is a direct consequence of the definition of the Zariski topology of X. □

From 1.3.3 we see that the algebra $k[X]$ completely determines X and its Zariski topology.

1.3.4. Exercises. (1) For any ideal I of $k[X]$ we have $\mathcal{I}_X(\mathcal{V}_X(I)) = \sqrt{I}$; for any subset Y of X we have $\mathcal{V}_X(\mathcal{I}_X(Y)) = \bar{Y}$.
(2) The map $\mathcal{I}_X$ defines a bijection of the family of Zariski-closed subsets of X onto the family of radical ideals of $k[X]$, with inverse $\mathcal{V}_X$.
(3) Let A be an affine k-algebra. Define a bijection of $\mathrm{Max}(A)$ onto the set of homomorphisms of k-algebras $A \to k$.
(4) Let X be an algebraic set.

(a) X is irreducible if and only if $k[X]$ is an integral domain (i.e., does not contain zero divisors $\neq 0$).

(b) X is connected if and only if the following holds: if $f \in k[X]$ and $f^2 = f$, $f \neq 0$ then $f = 1$.

(c) Let $X_1, \ldots, X_s$ be the irreducible components of X. If $X_i \cap X_j = \emptyset$ for $1 \leq i, j \leq s$, $i \neq j$, then there is an isomorphism $k[X] \to \bigoplus_{1 \leq i \leq s} k[X_i]$, defined by the restriction maps $k[X] \to k[X_i]$.

1.3.5. We shall have to consider locally defined functions on X. For this we need special open subsets of X, which we now introduce.

If $f \in k[X]$ put

$$D_X(f) = D(f) = \{x \in X \mid f(x) \neq 0\}.$$

This is an open set, namely the complement of $\mathcal{V}(fk[X])$. We have

$$D(fg) = D(f) \cap D(g),\ D(f^n) = D(f)\ (n \geq 1).$$

The $D(f)$ are called *principal open subsets* of X.

1.3.6. Lemma. *(i) If $f, g \in k[X]$ and $D(f) \subset D(g)$ then $f^n \in gk[X]$ for some $n \geq 1$;*
(ii) The principal open sets form a basis of the topology of X.

Using 1.1.4 (3) we see that $D(f) \subset D(g)$ if and only if $\sqrt{(fk[X])} \subset \sqrt{(gk[X])}$, which implies (i). (ii) is equivalent to the statement that every closed set in X is an intersection of sets of the form $\mathcal{V}_X(fk[X])$. This is obvious from the definitions. □