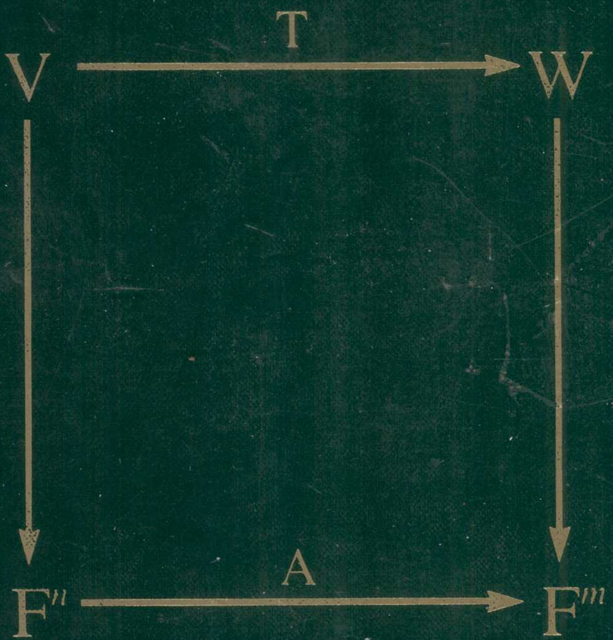
A SECOND COURSE IN LINEAR ALGEBRA



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A Second Course in Linear Algebra

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To Linda

Preface

For the past two years, I have been teaching a first-year graduate-level course in linear algebra and analysis. My basic aim in this course has been to prepare students for graduate-level work. This book consists mainly of the linear algebra in my lectures. The topics presented here are those that I feel are most important for students intending to do advanced work in such areas as algebra, analysis, topology, and applied mathematics.

Normally, a student interested in mathematics, engineering, or the physical sciences will take a one-term course in linear algebra, usually at the junior level. In such a course, a student will first be exposed to the theory of matrices, vector spaces, determinants, and linear transformations. Often, this is the first place where a student is required to do a mathematical proof. It has been my experience that students who have had only one such linear algebra course in their undergraduate training are ill prepared to do advanced-level work. I have written this book specifically for those students who will need more linear algebra than is normally covered in a one-term junior-level course.

This text is aimed at seniors and beginning graduate students who have had at least one course in linear algebra. The text has been designed for a one-quarter or semester course at the senior or first-year graduate level. It is assumed that the reader is familiar with such animals as functions, matrices, determinants, and elementary set theory. The presentation of the material in this text is deliberately formal, consisting mainly of theorems and proofs, very much in the spirit of a graduate-level course.

The reader will note that many familiar ideas are discussed in Chapter I. I urge the reader not to skip this chapter. The topics are familiar, but my approach, as well as the notation I use, is more sophisticated than a junior-level treatment. The material discussed in Chapters II–V is usually only touched upon (if at all) in a one-term course. I urge the reader to study these chapters carefully.

Having written five chapters for this book, I obviously feel that the reader should study all five parts of the text. However, time considerations often demand that a student or instructor do less. A shorter but adequate course could consist of Chapter I, Sections 1–6, Chapter II, Sections 1 and 2, and Chapters III and V. If the reader is willing to accept a few facts about extending scalars, then Chapters III, IV, and V can be read with no reference to Chapter II. Hence, a still shorter course could consist of Chapter I, Sections 1–6 and Chapters III and V.

It is my firm belief that any second course in linear algebra ought to contain material on tensor products and their functorial properties. For this reason, I urge the reader to follow the first version of a short course if time does not permit a complete reading of the text. It is also my firm belief that the basic linear algebra needed to understand normed linear vector spaces and real inner product spaces should not be divorced from the intrinsic topology and analysis involved. I have therefore presented the material in Chapter IV and the first half of Chapter V in the same spirit as many analysis texts on the subject. My original lecture notes on normed linear vector spaces and (real) inner product spaces were based on Loomis and Sternberg's classic text *Advanced Calculus.* Although I have made many changes in my notes for this book, I would still like to take this opportunity to acknowledge my debt to these authors and their fine text for my current presentation of this material.

One final word about notation is in order here. All important definitions are clearly displayed in the text with a number. Notation for specific ideas (e.g., $\mathbb{N}$ for the set of natural numbers) is introduced in the main body of the text as needed. Once a particular notation is introduced, it will be used (with only a few exceptions) with the same meaning throughout the rest of the text. A glossary of notation has been provided at the back of the book for the reader's convenience.

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September 1987

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Chapter I

Linear Algebra

1. DEFINITIONS AND EXAMPLES OF VECTOR SPACES

In this book, the symbol F will denote an arbitrary field. A field is defined as follows:

Definition 1.1: A nonempty set F together with two functions $(x, y) \to x + y$ and $(x, y) \to xy$ from $F \times F$ to F is called a *field* if the following nine axioms are satisfied:

F1. $x + y = y + x$ for all $x, y \in F$.

F2. $x + (y + z) = (x + y) + z$ for all $x, y, z \in F$.

F3. There exists a unique element $0 \in F$ such that $x + 0 = x$ for all $x \in F$.

F4. For every $x \in F$, there exists a unique element $-x \in F$ such that $x + (-x) = 0$.

F5. $xy = yx$ for all $x, y \in F$.

F6. $x(yz) = (xy)z$ for all $x, y, z \in F$.

F7. There exists a unique element $1 \neq 0$ in F such that $x1 = x$ for all $x \in F$.

F8. For every $x \neq 0$ in F, there exists a unique $y \in F$ such that $xy = 1$.

F9. $x(y + z) = xy + xz$ for all $x, y, z \in F$.

Strictly speaking a field is an ordered triple $(F, (x, y) \to x + y, (x, y) \to xy)$ satisfying axioms F1–F9 above. The map from $F \times F \to F$ given by $(x, y) \to x + y$ is called *addition*, and the map $(x, y) \to xy$ is called *multiplication*. When referring to some field $(F, (x, y) \to x + y, (x, y) \to xy)$, references to addition and multiplication are dropped from the notation, and the letter F is used to denote both the set and the two maps satisfying axioms F1–F9. Although this procedure is somewhat ambiguous, it causes no confusion in concrete situations. In our first example below, we introduce some notation that we shall use throughout the rest of this book.

Example 1.2: We shall let $\mathbb{Q}$ denote the set of rational numbers, $\mathbb{R}$, the set of real numbers, and $\mathbb{C}$, the set of complex numbers. With the usual addition and multiplication, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ are all fields with $\mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$. □

The fields in Example 1.1 are all infinite in the sense that the cardinal number attached to the underlying set in question is infinite. Finite fields are very important in linear algebra as well. Much of coding theory is done over finite algebraic extensions of the field $\mathbb{F}_p$ described in Example 1.3 below.

Example 1.3: Let $\mathbb{Z}$ denote the set of integers with the usual addition $x + y$ and multiplication xy inherited from $\mathbb{Q}$. Let p be a positive prime in $\mathbb{Z}$ and set $\mathbb{F}_p = \{0, 1, \ldots, p - 1\}$. $\mathbb{F}_p$ becomes a (finite) field if we define addition $\oplus$ and multiplication $\cdot$ modulo p. Thus, for elements $x, y \in \mathbb{F}_p$, there exist unique integers $k, z \in \mathbb{Z}$ such that $x + y = kp + z$ with $z \in \mathbb{F}_p$. We define $x \oplus y$ to be z. Similarly, $x \cdot y = w$ where $xy = k'p + w$ and $0 \leqslant w < p$.

The reader can easily check that $(\mathbb{F}_p, \oplus, \cdot)$ satisfies axioms F1–F9. Thus, $\mathbb{F}_p$ is a finite field of cardinality p. □

Except for some results in Section 7, the definitions and theorems in Chapter I are completely independent of the field F. Hence, we shall assume that F is an arbitrary field and study vector spaces over F.

Definition 1.4: A vector space V over F is a nonempty set together with two functions, $(\alpha, \beta) \to \alpha + \beta$ from $V \times V$ to V (called addition) and $(x, \alpha) \to x\alpha$ from $F \times V$ to V (called scalar multiplication), which satisfy the following axioms:

V1. $\alpha + \beta = \beta + \alpha$ for all $\alpha, \beta \in V$.
V2. $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$ for all $\alpha, \beta, \gamma \in V$.
V3. There exists an element $0 \in V$ such that $0 + \alpha = \alpha$ for all $\alpha \in V$.
V4. For every $\alpha \in V$, there exists a $\beta \in V$ such that $\alpha + \beta = 0$.
V5. $(xy)\alpha = x(y\alpha)$ for all $x, y \in F$, and $\alpha \in V$.
V6. $x(\alpha + \beta) = x\alpha + x\beta$ for all $x \in F$, and $\alpha, \beta \in V$.
V7. $(x + y)\alpha = x\alpha + y\alpha$ for all $x, y \in F$, and $\alpha \in V$.
V8. $1\alpha = \alpha$ for all $\alpha \in V$.

As with fields, we should make the comment that a vector space over F is really a triple $(V, (\alpha, \beta) \to \alpha + \beta, (x, \alpha) \to x\alpha)$ consisting of a nonempty set V together with two functions from $V \times V$ to V and $F \times V$ to V satisfying axioms V1–V8. There may be many different ways to endow a given set V with the structure of a vector space over F. Nevertheless, we shall drop any reference to addition and scalar multiplication when no confusion can arise and just use the notation V to indicate a given vector space over F.

If V is a vector space over F, then the elements of V will be called *vectors* and the elements of F *scalars*. We assume the reader is familiar with the elementary arithmetic in V, and, thus, we shall use freely such expressions as $-\alpha$, $\alpha - \beta$, and $\alpha_1 + \cdots + \alpha_n$ when dealing with vectors in V. Let us review some well-known examples of vectors spaces.

Example 1.5: Let $\mathbb{N} = \{1, 2, 3, \ldots\}$ denote the set of natural numbers. For each $n \in \mathbb{N}$, we have the vector space $F^n = \{(x_1, \ldots, x_n) \mid x_i \in F\}$ consisting of all n-tuples of elements from F. Vector addition and scalar multiplication are defined componentwise by $(x_1, \ldots, x_n) + (y_1, \ldots, y_n) = (x_1 + y_1, \ldots, x_n + y_n)$ and $x(x_1, \ldots, x_n) = (xx_1, \ldots, xx_n)$. In particular, when $n = 1$, we see F itself is a vector space over F. □

If A and B are two sets, let us denote the set of functions from A to B by B^A. Thus, $B^A = \{f: A \to B \mid f \text{ is a function}\}$. In Example 1.5, F^n can be viewed as the set of functions from $\{1, 2, \ldots, n\}$ to F. Thus, $\alpha = (x_1, \ldots, x_n) \in F^n$ is identified with the function $g_\alpha \in F^{\{1, \ldots, n\}}$ given by $g_\alpha(i) = x_i$ for $i = 1, \ldots, n$. These remarks suggest the following generalization of Example 1.5.

Example 1.6: Let V be a vector space over F and A an arbitrary set. Then the set V^A consisting of all functions from A to V becomes a vector space over F when we define addition and scalar multiplication pointwise. Thus, if $f, g \in V^A$, $f + g$ is the function from A to V defined by $(f + g)(a) = f(a) + g(a)$ for all $a \in A$. For $x \in F$ and $f \in V^A$, xf is defined by $(xf)(a) = x(f(a))$. □

If A is a finite set of cardinality n in Example 1.6, then we shall shorten our notation for the vector space V^A and simply write V^n. In particular, if $V = F$, then $V^n = F^n$ and we recover the example in 1.5.

Example 1.7: We shall denote the set of $m \times n$ matrices (a_{ij}) with coefficients $a_{ij} \in F$ by $M_{m \times n}(F)$. The usual addition of matrices $(a_{ij}) + (b_{ij}) = (a_{ij} + b_{ij})$ and scalar multiplication $x(a_{ij}) = (xa_{ij})$ make $M_{m \times n}(F)$ a vector space over F. □

Note that our choice of notation implies that F^n and $M_{1 \times n}(F)$ are the same vector space. Although we now have two different notations for the same vector space, this redundancy is useful and will cause no confusion in the sequel.

Example 1.8: We shall let F[X] denote the set of all polynomials in an indeterminate X over F. Thus, a typical element in F[X] is a finite sum of the form $a_nX^n + a_{n-1}X^{n-1} + \cdots + a_0$. Here $n \in \mathbb{N} \cup \{0\}$, and $a_0, \ldots, a_n \in F$. The usual notions of adding two polynomials and multiplying a polynomial by a constant, which the reader is familiar with from the elementary calculus, make sense over any field F. These operations give F[X] the structure of a vector space over F. □

Many interesting examples of vector spaces come from analysis. Here are some typical examples.

Example 1.9: Let I be an interval (closed, open, or half open) in $\mathbb{R}$. We shall let C(I) denote the set of all continuous, real valued functions on I. If $k \in \mathbb{N}$, we shall let $C^k(I)$ denote those $f \in C(I)$ that are k-times differentiable on the interior of I. Then $C(I) \supseteq C^1(I) \supseteq C^2(I) \supseteq \cdots$. These sets are all vector spaces over $\mathbb{R}$ when endowed with the usual pointwise addition $(f + g)(x) = f(x) + g(x)$, $x \in I$, and scalar multiplication $(yf)(x) = y(f(x))$. □

Example 1.10: Let $A = [a_1, b_1] \times \cdots \times [a_n, b_n] \subseteq \mathbb{R}^n$ be a closed rectangle. We shall let $\mathscr{R}(A)$ denote the set of all real valued functions on A that are Riemann integrable. Clearly $\mathscr{R}(A)$ is a vector space over $\mathbb{R}$ when addition and scalar multiplication are defined as in Example 1.9. □

We conclude our list of examples with a vector space, which we shall study carefully in Chapter III.

Example 1.11: Consider the following system of linear differential equations:

$$\begin{aligned} f_1' &= a_{11}f_1 + \cdots + a_{1n}f_n \\ &\vdots \\ f_n' &= a_{n1}f_1 + \cdots + a_{nn}f_n \end{aligned}$$

Here $f_1, \ldots, f_n \in C^1(I)$, where I is some open interval in $\mathbb{R}$. f_i' denotes the derivative of f_i, and the a_{ij} are scalars in $\mathbb{R}$. Set $A = (a_{ij}) \in M_{n \times n}(\mathbb{R})$. A is called the *matrix* of the system. If B is any matrix, we shall let B^t denote the transpose of B. Set $f = (f_1, \ldots, f_n)^t$. We may think of f as a function from $\{1, \ldots, n\}$ to $C^1(I)$, that is, $f \in C^1(I)^n$. With this notation, our system of differential equations becomes $f' = Af$. The set of solutions to our system is $V = \{f \in C^1(I)^n \mid f' = Af\}$. Clearly, V is a vector space over $\mathbb{R}$ if we define addition and scalar multiplication componentwise as in Example 1.9. □

Now suppose V is a vector space over F. One rich source of vector spaces associated with V is the set of subspaces of V. Recall the following definition:

Definition 1.12: A nonempty subset W of V is a subspace of V if W is a vector space under the same vector addition and scalar multiplication as for V.

Thus, a subset W of V is a subspace if W is closed under the operations of V. For example, $C([a, b])$, $C^k([a, b])$, $\mathbb{R}[X]$, and $\mathscr{R}([a, b])$ are all subspaces of $\mathbb{R}^{[a,b]}$.

If we have a collection $\mathscr{S} = \{W_i | i \in \Delta\}$ of subspaces of V, then there are some obvious ways of forming new subspaces from $\mathscr{S}$. We gather these constructions together in the following example:

Example 1.13: Let $\mathscr{S} = \{W_i | i \in \Delta\}$ be an indexed collection of subspaces of V. In what follows, the indexing set Δ of $\mathscr{S}$ can be finite or infinite. Certainly the intersection, $\bigcap_{i\in\Delta} W_i$, of the subspaces in $\mathscr{S}$ is a subspace of V. The set of all finite sums of vectors from $\bigcup_{i\in\Delta} W_i$ is also a subspace of V. We shall denote this subspace by $\sum_{i\in\Delta} W_i$. Thus, $\sum_{i\in\Delta} W_i = \{\sum_{i\in\Delta} \alpha_i | \alpha_i \in W_i \text{ for all } i \in \Delta\}$. Here and throughout the rest of this book, if Δ is infinite, then the notation $\sum_{i\in\Delta} \alpha_i$ means that all α_i are zero except possibly for finitely many $i \in \Delta$. If Δ is finite, then without any loss of generality, we can assume $\Delta = \{1, \ldots, n\}$ for some $n \in \mathbb{N}$. (If $\Delta = \phi$, then $\sum_{i\in\Delta} W_i = (0)$.) We shall then write $\sum_{i\in\Delta} W_i = W_1 + \cdots + W_n$.

If $\mathscr{S}$ has the property that for every $i, j \in \Delta$ there exists a $k \in \Delta$ such that $W_i \cup W_j \subseteq W_k$, then clearly $\bigcup_{i\in\Delta} W_i$ is a subspace of V. □

In general, the union of two subspaces of V is not a subspace of V. In fact, if W_1 and W_2 are subspaces of V, then $W_1 \cup W_2$ is a subspace if and only if $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$. This fact is easy to prove and is left as an exercise. In our first theorem, we discuss one more important fact about unions.

Theorem 1.14: Let V be a vector space over an infinite field F. Then V cannot be the union of a finite number of proper subspaces.

Proof: Suppose $W_1, \ldots, W_n$ are proper subspaces of V such that $V = W_1 \cup \cdots \cup W_n$. We shall show that this equation is impossible. We remind the reader that a subspace W of V is proper if $W \neq V$. Thus, $V - W \neq \phi$ for a proper subspace W of V.

We may assume without loss of generality that $W_1 \not\subseteq W_2 \cup \cdots \cup W_n$. Let $\alpha \in W_1 - \bigcup_{i=2}^{n} W_i$. Let $\beta \in V - W_1$. Since F is infinite, and neither α nor β is zero, $\Delta = \{\alpha + x\beta | x \in F\}$ is an infinite subset of V. Since there are only finitely many subspaces W_i, there exists a $j \in \{1, \ldots, n\}$ such that $\Delta \cap W_j$ is infinite.

Suppose $j \in \{2, \ldots, n\}$. Then there exist two nonzero scalars $x, x' \in F$ such that $x \neq x'$, and $\alpha + x\beta, \alpha + x'\beta \in W_j$. Since W_j is a subspace, $(x' - x)\alpha = x'(\alpha + x\beta) - x(\alpha + x'\beta) \in W_j$. Since $x' - x \neq 0$, we conclude $\alpha \in W_j$. But this is contrary to our choice of $\alpha \notin W_2 \cup \cdots \cup W_n$. Thus, $j = 1$.

Now if $j = 1$, then again there exist two nonzero scalars $x, x' \in F$ such that $x \neq x'$, and $\alpha + x\beta, \alpha + x'\beta \in W_1$. Then $(x - x')\beta = (\alpha + x\beta) - (\alpha + x'\beta) \in W_1$. Since $x - x' \neq 0$, $\beta \in W_1$. This is impossible since β was chosen in $V - W_1$. We conclude that V cannot be equal to the union of $W_1, \ldots, W_n$. This completes the proof of Theorem 1.14. □

If F is finite, then Theorem 1.14 is false in general. For example, let $V = (\mathbb{F}_2)^2$. Then $V = W_1 \cup W_2 \cup W_3$, where $W_1 = \{(0, 0), (1, 1)\}$, $W_2 = \{(0, 0), (0, 1)\}$, and $W_3 = \{(0, 0), (1, 0)\}$.

Any subset S of a vector space V determines a subspace $L(S) = \cap\{W \mid W$ a subspace of V, $W \supseteq S\}$. We shall call L(S) the *linear span* of S. Clearly, L(S) is the smallest subspace of V containing S. Thus, in Example 1.13, for instance, $L(\bigcup_{i\in\Delta} W_i) = \sum_{i\in\Delta} W_i$.

Let $\mathscr{P}(V)$ denote the set of all subsets of V. If $\mathscr{S}(V)$ denotes the set of all subspaces of V, then $\mathscr{S}(V) \subseteq \mathscr{P}(V)$, and we have a natural function $L: \mathscr{P}(V) \to \mathscr{S}(V)$, which sends a subset $S \in \mathscr{P}(V)$ to its linear span $L(S) \in \mathscr{S}(V)$. Clearly, L is a surjective map whose restriction to $\mathscr{S}(V)$ is the identity. We conclude this section with a list of the more important properties of the function $L(\cdot)$.

Theorem 1.15: The function $L: \mathscr{P}(V) \to \mathscr{S}(V)$ satisfies the following poperties:

(a) For $S \in \mathscr{P}(V)$, L(S) is the subspace of V consisting of all finite linear combinations of vectors from S. Thus,

$$L(S) = \left\{ \sum_{i=1}^{n} x_i\alpha_i \mid x_i \in F,\ \alpha_i \in S,\ n \geqslant 0 \right\}$$

(b) If $S_1 \subseteq S_2$, then $L(S_1) \subseteq L(S_2)$.

(c) If $\alpha \in L(S)$, then there exists a finite subset $S' \subseteq S$ such that $\alpha \in L(S')$.

(d) $S \subseteq L(S)$ for all $S \in \mathscr{P}(V)$.

(e) For every $S \in \mathscr{P}(V)$, $L(L(S)) = L(S)$.

(f) If $\beta \in L(S \cup \{\alpha\})$ and $\beta \notin L(S)$, then $\alpha \in L(S \cup \{\beta\})$. Here $\alpha, \beta \in V$ and $S \in \mathscr{P}(V)$.

Proof: Properties (a)–(e) follow directly from the definition of the linear span. We prove (f). If $\beta \in L(S \cup \{\alpha\}) - L(S)$, then β is a finite linear combination of vectors from $S \cup \{\alpha\}$. Furthermore, α must occur with a nonzero coefficient in any such linear combination. Otherwise, $\beta \in L(S)$. Thus, there exist vectors $\alpha_1, \ldots, \alpha_n \in S$ and nonzero scalars $x_1, \ldots, x_n, x_{n+1} \in F$ such that $\beta = x_1\alpha_1 + \cdots + x_n\alpha_n + x_{n+1}\alpha$. Since $x_{n+1} \neq 0$, we can write α as a linear combination of β and $\alpha_1, \ldots, \alpha_n$. Namely, $\alpha = x_{n+1}^{-1}\beta - x_{n+1}^{-1}x_1\alpha_1 - \cdots - x_{n+1}^{-1}x_n\alpha_n$. Thus, $\alpha \in L(S \cup \{\beta\})$. □

EXERCISES FOR SECTION 1

(1) Complete the details in Example 1.3 and argue $(\mathbb{F}_p, \oplus, \cdot)$ is a field.

(2) Let $\mathbb{R}(X) = \{f(x)/g(x) \mid f, g \in \mathbb{R}[X]$ and $g \neq 0\}$ denote the set of rational functions on $\mathbb{R}$. Show that $\mathbb{R}(X)$ is a field under the usual definition of addition $f/g + h/k = (kf + gh)/gk$ and multiplication $(f/g)(h/k) = fh/gk$. $\mathbb{R}(X)$ is called the field of rational functions over $\mathbb{R}$. Does F(X) make sense for any field F?

(3) Set $F = \{a + b\sqrt{-5} \mid a, b \in \mathbb{Q}\}$. Show that F is a subfield of $\mathbb{C}$, that is, F is a field under complex addition and multiplication. Show that $\{a + b\sqrt{-5} \mid a, b \text{ integers}\}$ is not a subfield of $\mathbb{C}$.

(4) Let I be an open interval in $\mathbb{R}$. Let $a \in I$. Let $V_a = \{f \in \mathbb{R}^I \mid f \text{ has a derivative at } a\}$. Show that V_a is a subspace of $\mathbb{R}^I$.

(5) The vector space $\mathbb{R}^{\mathbb{N}}$ is just the set of all sequences $\{a_i\} = (a_1, a_2, a_3, \ldots)$ with $a_i \in \mathbb{R}$. What are vector addition and scalar multiplication here?

(6) Show that the following sets are subspaces of $\mathbb{R}^{\mathbb{N}}$:
(a) $W_1 = \{\{a_i\} \in \mathbb{R}^{\mathbb{N}} \mid \lim_{i\to\infty} a_i = 0\}$.
(b) $W_2 = \{\{a_i\} \in \mathbb{R}^{\mathbb{N}} \mid \{a_i\} \text{ is a bounded sequence}\}$.
(c) $W_3 = \{\{a_i\} \in \mathbb{R}^{\mathbb{N}} \mid \sum_{i=1}^{\infty} a_i^2 < \infty\}$.

(7) Let $(a_1, \ldots, a_n) \in F^n - (0)$. Show that $\{(x_1, \ldots, x_n) \in F^n \mid \sum_{i=1}^{n} a_i x_i = 0\}$ is a proper subspace of F^n.

(8) Identify all subspaces of $\mathbb{R}^2$. Find two subspaces W_1 and W_2 of $\mathbb{R}^2$ such that $W_1 \cup W_2$ is not a subspace.

(9) Let V be a vector space over F. Suppose W_1 and W_2 are subspaces of V. Show that $W_1 \cup W_2$ is a subspace of V if and only if $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

(10) Consider the following subsets of $\mathbb{R}[X]$:
(a) $W_1 = \{f \in \mathbb{R}[X] \mid f(0) = 0\}$.
(b) $W_2 = \{f \in \mathbb{R}[X] \mid 2f(0) = f(1)\}$.
(c) $W_3 = \{f \in \mathbb{R}[X] \mid \text{the degree of } f \leqslant n\}$.
(d) $W_4 = \{f \in \mathbb{R}[X] \mid f(t) = f(1 - t) \text{ for all } t \in \mathbb{R}\}$.
In which of these cases is W_i a subspace of $\mathbb{R}[X]$?

(11) Let K, L, and M be subspaces of a vector space V. Suppose $K \supseteq L$. Prove Dedekind's modular law: $K \cap (L + M) = L + (K \cap M)$.

(12) Let $V = \mathbb{R}^3$. Show that $\delta_1 = (1, 0, 0)$ is not in the linear span of α, β, and γ where $\alpha = (1, 1, 1)$, $\beta = (0, 1, -1)$, and $\gamma = (1, 0, 2)$.

(13) If S_1 and S_2 are subsets of a vector space V, show that $L(S_1 \cup S_2) = L(S_1) + L(S_2)$.

(14) Let S be any subset of $\mathbb{R}[X] \subseteq \mathbb{R}^{\mathbb{R}}$. Show that $e^x \notin L(S)$.

(15) Let $\alpha_i = (a_{i1}, a_{i2}) \in F^2$ for $i = 1, 2$. Show that $F^2 = L(\{\alpha_1, \alpha_2\})$ if and only if the determinant of the 2×2 matrix $M = (a_{ij})$ is nonzero. Generalize this result to F^n.

(16) Generalize Example 1.8 to $n + 1$ variables $X_0, \ldots, X_n$. The resulting vector space over F is called the *ring of polynomials* in $n + 1$ variables (over F). It is denoted $F[X_0, \ldots, X_n]$. Show that this vector space is spanned by all monomials $X_0^{m_0}, \ldots, X_n^{m_n}$ as $(m_0, \ldots, m_n) \in (\mathbb{N} \cup \{0\})^{n+1}$.

(17) A polynomial $f \in F[X_0, \ldots, X_n]$ is said to be *homogeneous of degree d* if f is a finite linear combination of monomials $X_0^{m_0}, \ldots, X_n^{m_n}$ of degree d (i.e., $m_0 + \cdots + m_n = d$). Show that the set of homogeneous polynomials of degree d is a subspace of $F[X_0, \ldots, X_n]$. Show that any polynomial f can be written uniquely as a finite sum of homogeneous polynomials.

(18) Let $V = \{A \in M_{n \times n}(F) \mid A = A^t\}$. Show that V is a subspace of $M_{n \times n}(F)$. V is the subspace of symmetric matrices of $M_{n \times n}(F)$.

(19) Let $W = \{A \in M_{n \times n}(F) \mid A^t = -A\}$. Show that W is a subspace of $M_{n \times n}(F)$. W is the subspace of all skew-symmetric matrices in $M_{n \times n}(F)$.

(20) Let W be a subspace of V, and let $\alpha, \beta \in V$. Set $A = \alpha + W$ and $B = \beta + W$. Show that $A = B$ or $A \cap B = \varnothing$.

2. BASES AND DIMENSION

Before proceeding with the main results of this section, let us recall a few facts from set theory. If A is any set, we shall denote the cardinality of A by $|A|$. Thus, A is a finite set if and only if $|A| < \infty$. If A is not finite, we shall write $|A| = \infty$. The only fact from cardinal arithmetic that we shall need in this section is the following:

2.1: Let A and B be sets, and suppose $|A| = \infty$. If for each $x \in A$, we have some finite set $\Delta_x \subseteq B$, then $|A| \geqslant |\bigcup_{x \in A} \Delta_x|$.

A proof of 2.1 can be found in any standard text in set theory (e.g., [1]), and, consequently, we omit it.

A relation R on a set A is any subset of the crossed product $A \times A$. Suppose R is a relation on a set A. If $x, y \in A$ and $(x, y) \in R$, then we shall say x relates to y and write $x \sim y$. Thus, $x \sim y \Leftrightarrow (x, y) \in R$. We shall use the notation $(A, \sim)$ to indicate the composite notion of a set A and a relation $R \subseteq A \times A$. This notation is a bit ambiguous since the symbol $\sim$ has no reference to R in it. However, the use of $\sim$ will always be clear from the context. In fact, the only relation R we shall systematically exploit in this section is the inclusion relation $\subseteq$ among subsets of $\mathscr{P}(V)$ [V some vector space over a field F].

A set A is said to be partially ordered if A has a relation $R \subseteq A \times A$ such that (1) $x \sim x$ for all $x \in A$, (2) if $x \sim y$, and $y \sim x$, then $x = y$, and (3) if $x \sim y$, and $y \sim z$, then $x \sim z$. A typical example of a partially ordered set is $\mathscr{P}(V)$ together with the relation $A \sim B$ if and only if $A \subseteq B$. If $(A, \sim)$ is a partially ordered set, and $A_1 \subseteq A$, then we say A_1 is totally ordered if for any two elements $x, y \in A_1$, we have at least one of the relations $x \sim y$ or $y \sim x$. If $(A, \sim)$ is a partially ordered set, and $A_1 \subseteq A$, then an element $x \in A$ is called an upper bound for A_1 if $y \sim x$ for all $y \in A_1$. Finally, an element $x \in (A, \sim)$ is a maximal element of A if $x \sim y$ implies $x = y$.