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BLOW-UP IN NONLINEAR SOBOLEV TYPE EQUATIONS

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Preface

This monograph is devoted to the study of general problems on the global-in-time solvability and blow-up for a finite time of initial-value and initial-boundary-value problems for nonlinear equations of Sobolev type. Our studies together with an outstanding Russian mathematician S. A. Gabov, who prematurely died in 1989, stimulated further work in this direction.

Our study of the blow-up of solutions to pseudoparabolic nonlinear equations was considerably stimulated by the classical work of A. A. Samarskii, V. A. Galaktionov, S. P. Kurdyumov, and A. P. Mikhaylov *Blow-Up in Quasilinear Parabolic Equations*, which influenced the choice of many themes of our monograph.

We express a deep appreciation to V. P. Maslov, S. I. Pokhozhaev and N. N. Kalitkin for useful discussion of certain results presented in the monograph. We thank all participants of the scientific seminar "Nonlinear differential equations" and its supervisor Prof. I. A. Shishmaryov for their valuable comments on various sections of the book.

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Moscow, January 2011

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Chapter 0
Introduction

0.1 List of equations

First, we give a definition of Sobolev-type equations. Under a *Sobolev-type equation*, we mean an equation which is not resolved with respect to the partial derivative in time of the highest order. In our monograph, we use another term, *pseudoparabolic equations*, introduced by R. E. Showalter and T. W. Ting in [378]. *Pseudoparabolic equations* form a subclass of Sobolev-type equations with first-order derivatives in time. As in [378], we study only nonlinear, odd-order, pseudoparabolic equations, but, for completeness, we review results concerning nonlinear even-order equations.

0.1.1 One-dimensional pseudoparabolic equations

The Camassa–Holm equation

$$u_t - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}$$

describes the unidirectional propagation of shallow-water waves over a flat bottom (see [67–72, 200–202]). It is completely integrable (see [208]) and admits, in addition to smooth waves, a variety of travelling-wave solutions with singularities: peakons, cuspons, stumpons, and composite waves (see [258, 269, 270, 292]). This equation models wave breaking (see [95–101]).

The Degasperis–Procesi equation

$$u_t - u_{xxt} + 4uu_x = 3u_x u_{xx} + uu_{xxx}$$

models nonlinear shallow-water dynamics. It is completely integrable (see [208]) and has a variety of travelling-wave solutions including solitary-wave solutions, peakon solutions, and shock waves solutions (see [89, 90, 137, 259, 282, 291, 292]).

The Fornberg–Whitham equation

$$u_t - u_{xxt} + u_x + uu_x = uu_{xxx} + 3u_x u_{xx}$$

appeared in the study of the qualitative behavior of wave breaking (see [428]). It admits a wave of greatest height, as a peaked limiting form of the travelling-wave solution

$$u(x,t) = A \exp\left\{-\frac{1}{2}\left|x - \frac{4}{3}t\right|\right\}$$

(see [157]), where A is an arbitrary constant. It is not completely integrable (see [208]).

The Benjamin–Bona–Mahony (BBM) equation

$$u_t - u_{xxt} + u_x + uu_x = 0,$$

which is widely presented in the scientific literature; we note only the most significant works in which initial-value, initial-boundary-value, and periodic initial-boundary value problems were investigated: [2, 5, 24, 37, 39, 42–53, 78, 193, 297].

The modified Benjamin–Bona–Mahony (MBBM) equation

$$u_t - u_{xxt} + u_x + 3u^2 u_x = 0$$

has also been investigated by many authors (see, e.g., [38,66,273,310,343,344,438]).

The hyperelastic-rod wave equation

$$u_t - u_{xxt} + 3uu_x = \gamma(2u_x u_{xx} + uu_{xxx})$$

was considered in [208].

The generalized hyperelastic-rod wave equation

$$u_t - u_{xxt} + \partial_x(\varkappa u + \alpha u^2 + \gamma u^3) = \nu u_x u_{xx} + uu_{xxx}$$

is a generalization of the previous equation.

0.1.2 One-dimensional wave dispersive equations

The Benjamin–Bona–Mahony–Burgers (BBMB) equation

$$u_t - u_{xxt} + u_x + uu_x - \nu u_{xx} = 0$$

was considered, e.g., in [1, 37, 48–53, 78, 140, 178, 188, 296, 402, 441].

The generalized porous-media equation

$$u_t = \partial_x(u^\alpha u_x + u^\beta u_{xt}), \quad \alpha, \beta \geq 0,$$

which was studied, e.g., in [106].

The Rosenau–Burgers equation

$$u_{xxxxt} + u_t - \alpha u_{xx} + u^p u_x = 0.$$

The breaking result for this equation was obtained in [315, 316, 348–352]. Moreover, in [197, 296, 315, 316], the first terms of asymptotic expansions of solutions for large time were found. In addition, we note the paper [83, 86], where Galerkin approximations were considered.

0.1.3 Singular one-dimensional pseudoparabolic equations

In these equations, elliptic operators under the highest time derivative are not resolvable.

The Coleman–Duffin–Mizel equation

$$u_t + u_{xxt} - u_{xx} = 0$$

was considered in [91].

The Hoff equation

$$u_t + u_{xxt} = \alpha u + \beta u^3$$

was studied, e.g., in [199, 395–398]. The semigroup approach to the general theory of Sobolev-type singular equations was developed by G. A. Sviridyuk and V. E. Fyodorov. In an abstract form, degenerate pseudoparabolic equations were considered, e.g., in [148, 431–433].

The Korpusov–Pletner–Sveshnikov equation

$$u_t + u_{xxt} + \alpha u_{xx} + \beta(u^2)_{xx} = 0, \quad \alpha > 0,\ \beta > 0,$$

describes nonstationary processes in crystalline semiconductors.

The one-dimensional Oskolkov equation

$$u_t + u_{xxt} + u u_x + \nu u_{xx} = 0.$$

The one-dimensional Boussinesq equation

$$u_t + u_{xxt} + \nu(|u|^{p-2}u)_{xx} = 0$$

was considered in [130, 395, 398].

0.1.4 Multidimensional pseudoparabolic equations

The Barenblatt–Zheltov–Kochina equation

$$\frac{\partial}{\partial t}(\Delta u + cu) + \Delta u = 0, \quad c \in \mathbb{R}^1 \backslash \{0\},$$

describes nonstationary filtering processes in fissured-porous media. This equation can be rewritten in a more general form

$$\frac{\partial}{\partial t}\mathrm{A}(u) + \mathrm{B}(u) = 0,$$

where $\mathrm{A}(u)$ and $\mathrm{B}(u)$ are nonlinear elliptic operators. In the classical works [365–371, 374, 378] R. E. Showalter and T. W. Ting considered linear equations of this form.

The Showalter equation

$$\frac{\partial}{\partial t}(\Delta u + \operatorname{div}(|\nabla u|^{p-2}\nabla u) - u) + \alpha\Delta u + \alpha \operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0$$

and initial-value problems for it were considered in [368, 369] and some general results on the unique solvability for abstract pseudoparabolic equations were obtained.
The Showalter inclusion

$$\frac{\partial}{\partial t}\mathrm{A}(u) + \mathrm{B}(u) \ni f,$$

where $\mathrm{A}(u)$ and $\mathrm{B}(u)$ are maximal monotone elliptic operators, were studied in [113–120, 365–371, 374, 388–392]. Moreover, we mention the following works devoted to the study of initial-value and initial-boundary-value problems for multidimensional dissipative pseudoparabolic equations: [12, 31–36, 39–42, 57–61, 64, 65, 77–79, 93, 94, 103–106, 113–120, 143–149, 179, 195–197, 217, 251–255, 306–309, 315, 316, 364]. We also note the monograph of H. Gajewski, K. Gröger, and K. Zacharias [168], in which various aspects of the local solvability of pseudoparabolic equations are considered. The central part of this monograph is devoted to the study of operator and operator-differential equations. For pseudoparabolic operator-differential equations, the aspects of $\mathbb{C}$- and $\mathbb{L}^2$-solvability are analyzed. The basis of finite-dimensional approximate methods, especially the Galerkin method, is considered.

We note that the method of construction of asymptotical expansions for large time for a wide class of nonlinear evolutionary equations developed in works of N. Hayashi, I. A. Shishmaryov, P. I. Naumkin, and E. I. Kaikina (see [195–197, 306–309]) can be applied to the study of the asymptotical behavior for large time for pseudoparabolic equations. Specifically, in [364] the asymptotic behavior of solutions of the Cauchy problem for the following dissipative pseudoparabolic equation was obtained.
The semiconductor equation

$$\frac{\partial}{\partial t}(\Delta u - u) + \Delta u + \alpha u^3 = 0, \quad \alpha \in \mathbb{R}.$$

was obtained in [236]; it describes nonstationary processes in crystalline semiconductors.

The generalized Boussinesq nonlinear equation

$$u_t - \Delta\psi(u) - \Delta u_t + q(u) = 0.$$

Based on the comparison principle, A. I. Kozhanov [243] proved the solvability of the first boundary-value problem and the occurrence of the blow-up. The blow-up of positive solutions was proved and existence/nonexistence theorems were obtained.
The multidimensional Benjamin–Bona–Mahony–Burgers equation

$$\frac{\partial}{\partial t}(\Delta u - u) + (\lambda, \nabla)(u + u^2) + \Delta u = 0$$

was considered in [217, 429, 441].

The linear Rossby wave equation

$$\frac{\partial}{\partial t}\Delta_2 u + \beta(y)\frac{\partial u}{\partial x} = 0, \quad \Delta_2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

is a linear approximation in the β-plane of the two-dimensional Rossby wave equation (see [164, 165]), where the axes Ox and Oy are directed to the east and to the north, respectively, and $\beta = \beta(y)$ is the Coriolis parameter.

The Kadomtsev–Petviashvilli equation

$$\frac{\partial}{\partial t}(\Delta_2 u - u) + \frac{\partial u}{\partial x} + u\frac{\partial u}{\partial x} + J(u, \Delta_2 u) = 0, \quad J(a,b) = a_x b_y - a_y b_x,$$

is a nonlinear generalization of the two-dimensional equation of Rossby waves (see, e.g., [400]).

The three-dimensional Camassa–Holm equation

$$\frac{\partial}{\partial t}(\Delta \mathbf{u} - \mathbf{u}) + \nu\Delta(\Delta \mathbf{u} - \mathbf{u}) + \mathbf{u} \times (\nabla \times (\Delta \mathbf{u} - \mathbf{u})) = \nabla p,$$

is the viscous version of the three-dimensional Camassa–Holm equations; it was considered in [202].

0.1.5 New nonlinear pseudoparabolic equations with sources

Now we list some equations obtained in our works [233–240].

The generalized Benjamin–Bona–Mahony–Burgers equation

$$\frac{\partial}{\partial t}(\Delta u - u - |u|^{q_1} u) + \frac{\partial u}{\partial x_1} + u\frac{\partial u}{\partial x_1} + \Delta u + |u|^{q_2} u = 0,$$

where $x = (x_1, x_2, x_3) \in \Omega \subset \mathbb{R}^3$, $q_1, q_2 > 0$,

$$\Delta \equiv \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}.$$

The nonlocal pseudoparabolic equation

$$\frac{\partial}{\partial t}(\Delta u - u) + \Delta u\left[\int_\Omega |\nabla u|^2\, dx\right]^q = 0, \quad q > -1.$$

The generalized Rosenau–Burgers equation

$$\frac{\partial}{\partial t}(-\Delta^2 u + \Delta u + \operatorname{div}(|\nabla u|^{p_1 - 2}\nabla u)) + \Delta u - \operatorname{div}(|\nabla u|^{p_2 - 2}\nabla u) = 0,$$

where

$$\Delta^2 \equiv \Delta\Delta, \quad p_1, p_2 \geq 2.$$

The spin-wave equation

$$\frac{\partial}{\partial t}(-\Delta^2 u + \Delta u + \operatorname{div}(|\nabla u|^{p-2}\nabla u)) + \Delta u - \operatorname{div}(|\nabla u|^2 \nabla u)$$
$$+ \alpha_1 \frac{\partial}{\partial x_1}\left(\frac{\partial u}{\partial x_2}\frac{\partial u}{\partial x_3}\right) + \alpha_2 \frac{\partial}{\partial x_2}\left(\frac{\partial u}{\partial x_3}\frac{\partial u}{\partial x_1}\right) + \alpha_3 \frac{\partial}{\partial x_3}\left(\frac{\partial u}{\partial x_1}\frac{\partial u}{\partial x_2}\right) = 0,$$

where $|\alpha_1| + |\alpha_2| + |\alpha_3| > 0$, $\alpha_1 + \alpha_2 + \alpha_3 = 0$, $p \geq 2$.

0.1.6 Model nonlinear equations of even order

For completeness, we list some model, nonlinear, even-order equations although we will not study them in what follows.

One-dimensional nonlinear equations

The Longern wave equation

$$\frac{\partial^2}{\partial t^2}(u_{xx} - \alpha u + \beta u^2) + u_{xx} = 0, \quad \alpha > 0,\ \beta > 0,$$

which describes electric signals in telegraph lines [281]. Moreover, in active nonlinear media, the following equations hold.

The Rabinowitz wave equations with nonlinear damping

$$\frac{\partial^2}{\partial t^2}(u_{xx} - u) + \frac{\partial^2}{\partial t \partial x}\left(u_x - (u_x)^2\right) + u_{xx} = 0,$$
$$\frac{\partial^2}{\partial t^2}(u_{xx} - u) + \frac{\partial}{\partial t}\left(u - u^2\right) + u_{xx} = 0,$$

which describe electric signals in telegraph lines on the basis of the tunnel diode (see [340, 341]). Note that in our book [230], sufficient conditions of the blow-up for the corresponding initial-boundary-value problems were obtained.

The nonlinear telegraph equation with nonlinear damping

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = \alpha \frac{\partial^4 u}{\partial x^2 \partial t^2} + \beta \frac{\partial^3}{\partial x^2 \partial t}\left(\frac{u^3}{3} - u\right), \quad \alpha > 0, \quad \beta > 0,$$

which was considered in [62, 223].

The Pochhammer–Chree equation

$$u_{tt} - u_{xxtt} - (a_1 u + a_3 u^3 + a_5 u^5)_{xx} = 0$$

was considered in [442–444].