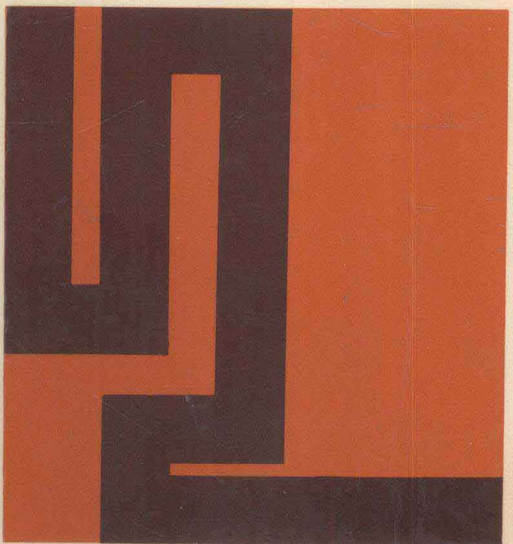
Sets Relations Functions

Samuel Selby and Leonard Sweet



SETS—RELATIONS—FUNCTIONS

Second Edition

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Preface

A primary aim of all recommendations proposed by various professionally recognized committees, such as the School Mathematics Study Group (SMSG) and the Committee on the Undergraduate Program in Mathematics (CUPM), is to make an integrated course in calculus and analytic geometry a standard beginning course for all college freshmen. To institute such a change will require a rearrangement in the academic experiences of the high school student and, in many instances, a retraining of elementary and secondary teaching personnel.

Concepts previously reserved for the more sophisticated courses in mathematics will have to be reshuffled and placed in courses at lower levels. Since this book was first published in 1963, many important changes have taken place at all levels of the mathematics spectrum. These changes are no longer experimental in character but are reflected in revised curricula, new programs, and new textbooks that involve the experiences of many teachers of mathematics. This text, as initially prepared, was written to enable students and teachers to meet the changing situation in mathematics with greater confidence and interest. This continues to be a primary objective of the authors in this second edition.

The concept of a set is becoming more important than ever to the elementary teacher who desires to understand the principles of arithmetic from kindergarten and upward in a more effective fashion. To the secondary school teacher, familiarity with sets provides a common medium for presenting algebra and geometry in a better fashion. The ideas concerned with the use of inequality, absolute value, the number system viewed as a structure, postulational proof, and the ordered pair and its implications for explaining more adequately the concept of a function versus a relation, and many other such notions, are basic essentials in the mathematical background of such a professional person.

As in the previous edition, this new revision is not an "end-all." The objective of the authors is to present those ideas and symbolisms from set theory that will lead the interested reader to a keener insight into what has already been experienced in his mathematical background. It is the belief of the authors that the utilization of the language and concepts of set theory—which is not an end in itself—should prove fruitful in terms of a more thorough understanding, appreciation, and enthusiasm for other areas in mathematics.

The usual criticism leveled at corresponding texts is either that too much is included on set theory so that the reader is lost in its complexities or that, after an initial presentation, no further application of set theory is made to actual problem material in algebra, geometry, and other mathematical experiences of the reader. It is hoped that this text avoids such pitfalls.

Among the major changes in this edition are the following:

1. Completely new sections have been included covering
a. Mathematical induction
b. Finite and infinite sets
c. Quantifiers
d. Completeness property
e. Partition and equivalence relations
f. Mappings and images

2. Several sections dealing with absolute value have been extended and enlarged to include the neighborhood concept.

3. A completely new section on logic has been added in Chapter 1 and enlarged in Chapter 5 for the study of valid arguments and the algebra of propositions.

4. The treatment of functions, inverse functions, operations with functions, and functions as mappings has been amplified.

5. Many new problems and illustrations have been added to provide additional practice and further insight into the conceptual material of the text. The book now includes approximately 250 worked-out examples and 180 graphical representations. The exercise material involves approximately 1800 problems, for which an answer section to selected problems is included. On an overall basis, the increase in these various categories is approximately 20 percent.

Each new idea, as in the past edition, is presented with several illustrations, and if it is a basic concept, it is reemphasized before being integrated. As a consequence, the authors may be accused of repetition, but from a pedagogical point of view, they stand ready to accept this criticism.

Scattered throughout the text are various supplementary exercises which have been enhanced and which are referred to as projects. These project exercises were inspired by the authors' experiences involving groups of elementary and secondary teachers participating in several National Science Foundation Institute programs held at the University of Akron since 1960.

The expository material as presented in this second edition may be used either for its own sake, as a text in a high school class or in a freshman college class, or as a supplementary reference book in conjunction with other standard high school or college texts in mathematics. It should certainly be serviceable as text material for teachers of both the elementary and secondary levels where local school systems are conducting in-service programs or National Science Foundation Institute programs.

The authors are particularly grateful for comments and suggestions received from their colleagues at the University of Akron and from other instructors and students. We acknowledge especially the cooperation received from Professor George Szoke, who permitted the inclusion of certain problem material pertaining to the neighborhood of a point. We are indebted to Mrs. Ruth Sweet for her patient handling of the typing and retyping which was so essential in preparing the final draft of this revised edition.

The authors take full responsibility for any shortcomings which may be found in this material. They welcome constructive criticism from its users.

Samuel Selby
Leonard Sweet

List of Symbols

$\in$	is an element of, belongs to	2
$\notin$	is not an element of, does not belong to	2
N	set of natural numbers	2
P	set of primes	2
I	set of integers	2
F	set of rational numbers	2
R_e	set of real numbers	6
C	set of complex numbers	7
$\mid$ or $:$	such that, for which	6
$\leftrightarrow$	is in one-to-one correspondence with	10
$=$	is equal to	15
$C(x)$	defining condition involving x	6
$\neq$	is not equal to	15
$\subseteq$	is included in	16
$\nsubseteq$	is not included in	16
$\subset$	is a proper subset of	17
$\not\subset$	is not a proper subset of	17
$\supset$	is a superset of	17
$<$	is less than	101
$>$	is greater than	101
$\leq$	is less than or equal to	101
$\geq$	is greater than or equal to	101
$\rightarrow$	maps into	9
U	universe	18
$\emptyset$	null set or empty set	4
$n(A)$	number of elements in set A	36
$\cup$	union	27
$\cap$	intersection	26
A'	complement of set A	27
$\wedge$	and	26
$\vee$	or (inclusive)	27
$\underline{\vee}$	or (exclusive)	27
$\Rightarrow$	implies	58
$\Leftrightarrow$	is logically equivalent to	64
$\sim$	not	52
2^P	power set of P	22
$\aleph_0$	aleph-null	48
$\forall x$	for all x	67
$\exists x$	for some x	67
$N_b a$	neighborhood of a of radius b	117
$N_b^* a$	deleted neighborhood of a of radius b	117
$\text{glb}(K)$	greatest lower bound of K	132
$\text{lub}(K)$	least upper bound of K	132

$\mathrm{cl}(a)$	equivalence class	162
$[x]$	greatest integer value	219
$\|a\|$	absolute value of a	109
$[a,b]$	$\{x \in R_e \mid a \leq x \leq b\}$	115
$[a,b)$	$\{x \in R_e \mid a \leq x < b\}$	115
$(a,b]$	$\{x \in R_e \mid a < x \leq b\}$	115
(a,b)	$\{x \in R_e \mid a < x < b\}$	115
(x,y)	ordered pair $\Leftrightarrow$ coordinates of a point in the plane	146
$A \times B$	cartesian product of set A with set B	148
$P(x,y)$	defining condition involving x and y	155
R	relation	154
R'	complement of the relation R	165
R^{-1}	inverse of the relation R	166
f	function	209
f^{-1}	inverse of the function f	248
$f(x)$	value of the function f at x	210
D^*	domain	156
R^*	range	156
$f + g$	sum function	243
$f - g$	difference function	243
fg	product function	243
$\frac{f}{g}$	quotient function	243
$f \circ g$	composite function	252
$*$ (in $a * b$)	operation	268
$\equiv$	is congruent modulo	319
$\cong$	is congruent to	159

GREEK ALPHABET

Greek letter	Greek name	English equivalent	Greek letter	Greek name	English equivalent
$\mathrm{A}\ \alpha$	Alpha	a	$\mathrm{N}\ \nu$	Nu	n
$\mathrm{B}\ \beta$	Beta	b	$\Xi\ \xi$	Xi	x
$\Gamma\ \gamma$	Gamma	g	$\mathrm{O}\ o$	Omicron	ŏ
$\Delta\ \delta$	Delta	d	$\Pi\ \pi$	Pi	p
$\mathrm{E}\ \epsilon$	Epsilon	ĕ	$\mathrm{P}\ \rho$	Rho	r
$\mathrm{Z}\ \zeta$	Zeta	z	$\Sigma\ \sigma\ s$	Sigma	s
$\mathrm{H}\ \eta$	Eta	ē	$\mathrm{T}\ \tau$	Tau	t
$\Theta\ \theta\ \vartheta$	Theta	th	$\Upsilon\ \upsilon$	Upsilon	u
$\mathrm{I}\ \iota$	Iota	i	$\Phi\ \phi\ \varphi$	Phi	ph
$\mathrm{K}\ \kappa$	Kappa	k	$\mathrm{X}\ \chi$	Chi	ch
$\Lambda\ \lambda$	Lambda	l	$\Psi\ \psi$	Psi	ps
$\mathrm{M}\ \mu$	Mu	m	$\Omega\ \omega$	Omega	ō

Contents

1
The Vocabulary and Symbolism of Sets

1.1 INTRODUCTION

Certain ideas in mathematics, because of their scope and simplicity, constitute reservoirs of untold richness. The theory of sets, which is one such idea, was developed by Georg Cantor (1845–1918). Few concepts in the past hundred years have had as great an impact on mathematics as has the notion of a set. Set theory has contributed a foundation which clarifies and unifies the mathematics already developed. It provides a language and a symbolism which make it possible to synthesize the old and the new, to examine familiar concepts, and to view new and exciting milestones along the mathematical highway. To reach the first milestone on the highway, a familiarity must be established with the vocabulary and symbolism of set theory—the objective of Chapter 1.

1.2 CONCEPT OF A SET

Intuitively, a set is any well-defined collection of objects. Other words, such as "collection," "class," and "aggregate," are used synonymously with the term "set." "Well-defined" means that it is possible in principle to determine whether an object is a member of a set or not. For example, the "set of the five greatest living statesmen" is not well-defined. This is because the criteria or standards as to what determines a great living statesman are not commonly agreed upon by everyone. Given the name of a particular statesman, we would have difficulty in determining definitely whether this individual is a member of or is not a member of the set. However, with the set of 50 states of the United States, no difficulty would be experienced in determining definitely whether any given object is or is not contained in this set.

The individual objects that belong to a set are called its elements. If capital letters $A, B, C, \ldots$ denote sets and small letters $a, b, c, \ldots$ represent elements, then the notation $a \in A$ is read "a belongs to A" or "a is an element of A." $b \notin B$ is read "b does not belong to B." The symbols $\in$ (belongs to) and $\notin$ (does not belong to) are referred to as the "membership relations." The notation $x_1, x_2, x_3, \ldots, x_n \in A$ means that each $x_i \in A$.

Sets may also be collections of sets. For example, the set of baseball teams in the National League is a set of teams where each team is an element. Further, each player is an element of the set constituting the team on which he plays.

The following examples illustrate the concept of set and the membership relation.

Example 1. If G = the set of vowels in the English alphabet, then e $\in G$ but r $\notin G$.

Example 2. If B = the set of months beginning with the letter J, then January $\in B$ but May $\notin B$.

Example 3. Let N = the set of natural numbers (counting integers, excluding zero). Hence $3 \in N$ and $11 \in N$, but $\frac{3}{4} \notin N$ and $0 \notin N$.

Example 4. Let P = the set of natural primes (a natural number is a prime if it has only two distinct natural-number divisors, itself and 1). Hence $2 \in P$ and $5 \in P$, but $1 \notin P$ and $8 \notin P$.

Example 5. Let I = the set of integers (positive and negative integers and zero). Then $0 \in I$, $3 \in I$, and $-5 \in I$; but $\frac{2}{3} \notin I$ and $\frac{9}{4} \notin I$.

Example 6. Let F = the set of rational numbers (a number is rational if it can be expressed as the quotient p/q, of two integers p and q where $q \neq 0$). Then $\frac{3}{5} \in F$, $\frac{4}{3} \in F$, and -5 or $-5/1 \in F$; but $\sqrt{3} \notin F$, sin $12° \notin F$, and log $17 \notin F$.

Example 7. If N = the set of natural numbers, P = the set of natural primes, I = the set of integers, and F = the set of rational numbers, then the membership relation ($\in$ or $\notin$) for each of the numbers 0, 1, -2, $\frac{3}{4}$, $-\frac{9}{4}$, $\frac{10}{6}$, -8, π, $3\frac{1}{3}$, 5, sin 30°, and cos ($\pi/10$) is indicated in Table 1.

Table 1

	N	P	I	F
0	$\notin$	$\notin$	$\in$	$\in$
1	$\in$	$\notin$	$\in$	$\in$
-2	$\notin$	$\notin$	$\in$	$\in$
$\frac{3}{4}$	$\notin$	$\notin$	$\notin$	$\in$
$-\frac{9}{4}$	$\notin$	$\notin$	$\notin$	$\in$
-8	$\notin$	$\notin$	$\in$	$\in$
$\frac{10}{6}$	$\notin$	$\notin$	$\notin$	$\in$
$3\frac{1}{3}$	$\notin$	$\notin$	$\notin$	$\in$
5	$\in$	$\in$	$\in$	$\in$
sin 30°	$\notin$	$\notin$	$\notin$	$\in$
$\cos\left(\frac{\pi}{10}\right)$	$\notin$	$\notin$	$\notin$	$\notin$
π	$\notin$	$\notin$	$\notin$	$\notin$

Example 8. Let T = the set of integers satisfying the equation $2x - 7 = 5$. Hence $6 \in T$. Note that 6 is the only element in T.

EXERCISE 1

1. A set must be a well-defined collection of objects. Which of the following objects form sets according to this definition?

a. The set of the three greatest musical compositions
b. The set of all months of the year beginning with the letter D
c. The set of the 10 greatest living Americans
d. The set of words appearing in this book
e. The set of the five most talented actors

2. Let Q denote the set of all the quadrilaterals of plane geometry. Using the connectives $\in$ and $\notin$, indicate the membership relation for each of the following figures.

Example: A triangle t — *Answer:* $t \notin Q$

a. A rhombus r — *b.* A square □ — *c.* A parallelogram p
d. A hexagon h — *e.* A rectangle ▯ — *f.* A pentagon g
g. A trapezoid z — *h.* A circle c

3. Which of the following sets have elements that are also sets?

a. The set of football teams in the National Football League
b. The Cleveland Symphony Orchestra

c. The American Federation of Labor
d. The United Nations
e. The set of all counties in the United States

1.3 FINITE AND INFINITE SETS

If a set is finite, then it contains exactly n different elements, where n is some natural number. If a set is not finite, then it is infinite.

The natural numbers 1, 2, 3, . . . , n, . . . represent an infinite set. Other examples of infinite sets are:

1. The set of all circles in a plane
2. The set of odd primes
3. The set of integers divisible by 3
4. The set of rational numbers greater than zero
5. The set of points on a line

If a set is finite, then it is conceptually possible to determine the number of elements that belong to the set. Examples of finite sets are:

1. The set of months of the year (12 elements)
2. The set of letters in the English alphabet (26 elements)
3. The set of all books in the Library of Congress (more than 12 million elements)
4. The set of all the grains of sand on the beach at Atlantic City (though this set is very large, it contains a finite number of elements)
5. The set of even primes (one element)

For the purpose of generalization, a set containing no elements is defined as a finite set. A set containing no elements is called the null or empty set and is symbolized by the notation $\emptyset$ or $\{\quad\}$. Examples of the null set are:

1. The set of integers between 3 and 4
2. The set of students in your classroom whose birthplace was the planet Jupiter
3. The set of integers satisfying the equation $9x^2 - 4 = 0$
4. The set of points that are common to both squares shown in Fig. 1

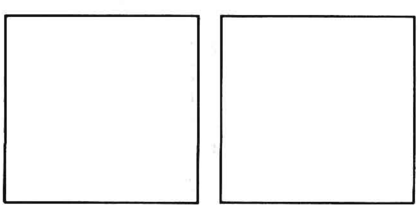

Fig. 1

EXERCISE 2

1. Give two examples of finite sets; two examples of infinite sets.

2. Give two descriptive examples of the empty set.

3. State whether each of the following sets is finite or infinite. When the set is finite, indicate the number of elements it possesses.

a. The set of letters in the word Massachusetts
b. The set of odd positive integers
c. The set of all positive two-digit integers
d. The set of integers satisfying the equation $x^2 - 5x - 6 = 0$
e. The set of rational numbers satisfying the equation $x^2 - 5 = 0$
f. The set of quadrilaterals
g. The set of primes greater than 2 and less than 75
h. The set of students in your school who have two heads
i. The set of all lines passing through a fixed point in a plane
j. The set of intersection points of two circles in a plane

1.4 DESCRIPTION OF SETS

Two methods used frequently to describe sets are the *tabulation method* and the *defining-property method.* The first, the tabulation method, enumerates or lists the individual elements, separates them by commas, and encloses them in braces.

Example 1. $\{3,2,5\}$ is the set whose elements are the numbers 3, 2, and 5. The order of listing is unimportant; the set $\{3,2,5\}$ is identical with the set $\{5,3,2\}$. In fact, there are six ways of listing this set: $\{3,2,5\}$, $\{3,5,2\}$, $\{5,3,2\}$, $\{5,2,3\}$, $\{2,5,3\}$, and $\{2,3,5\}$.

If a set contains only one element, it is called a unit set. $A = \{x\}$ is an example of a unit set whose only member is x. Note that $A = \{x\}$ is not the same as $A = x$.

Enumeration of an infinite set consists in listing a few elements followed by three dots. $N = \{1,2,3,4, \ldots\}$ is the notation for the infinite set of natural numbers. The same notation is used for a finite set, but the last element is always included. The set of natural numbers less than 1000 can be written as $M = \{1,2,3, \ldots ,998,999\}$. It should be noted that the enumeration procedure should be used with care, since it is possible that a partial enumeration can often lead to more than one interpretation.

Some sets cannot be described by an enumeration. A second method, which defines a property, is often more compact and convenient. For example, the set of rational numbers between 5 and 6 and the set of even integers between 1 and 25 are described by

$$F = \{\text{all rational numbers between 5 and 6}\}$$
$$E = \{\text{all even integers between 1 and 25}\}$$

This defining condition may take different forms. It may consist of descriptive words, symbols from mathematics, or a combination of both. The defining condition spells out specifically the requirements that an object must satisfy in order to belong to the set. The defining-property method provides, in a precise form, a test for membership of all the elements belonging to the set, which the tabulation method lacks. This is especially true with regard to those elements absent from a listing of the set.

Example 2. The set of one-digit primes could be represented by $A = \{2,3,5,7\}$ according to the tabulation method. By the defining-property method we have $A = \{x \mid x \text{ is a one-digit prime}\}$ or

$$A = \{x : x \text{ is a one-digit prime}\}$$

This is read "the set of all elements x such that x is a one-digit prime." The symbol $|$ or $:$ is read "such that" or "for which." The notation then takes the general form $\{x \mid \text{some defining condition about } x\}$ or $\{x \mid C(x)\}$ where $C(x)$ represents the defining condition involving x. Frequently, the notation is modified and written

$$A = \{x \in P \mid x \text{ is a one-digit number}\}$$

Here P is the set of all primes and A is now read "the set of all those elements x of P such that x is a one-digit number."

The letter x is called a variable or placeholder. It should be noted that any other desired symbol, such as y, z, a_1, a_2, $*$, $\triangle$, $\square$, or α, could be used to represent the variable. This symbol holds the place for any element of the set that is being defined. The elements of the defined set are referred to as the values of the variable. When a defined set consists of just a single value—for example, π—we call the variable a constant. Thus a variable may hold the place for either a finite number of values or an infinite number of values, depending upon the defined set.

The following sentences are *defining conditions* placed on the elements:

x is an even integer.
x is an integer such that $x + 7 = 8$.
x is a natural number such that $x^2 - 3x + 2 = 0$.
x is an integer greater than 5.
x is a natural number divisible by 3.

The following examples illustrate the use of defining conditions in describing sets.

Example 3. Let R_e denote the set of real numbers, i.e., the collection of all rational and irrational numbers. Irrational numbers are numbers