MATHEMATICS + PHYSICS

Lectures on Recent Results

Volume 3

Editor: L Streit

World Scientific
Singapore • New Jersey • London • Hong Kong

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Published by

World Scientific Publishing Co. Pte. Ltd.
P O Box 128, Farrer Road, Singapore 9128

USA office: World Scientific Publishing Co., Inc.
687 Hartwell Street, Teaneck, NJ 07666, USA

UK office: World Scientific Publishing Co. Pte. Ltd.
P O Box 379, London N12 7JS, England

MATHEMATICS + PHYSICS (VOLUME 3)

ISBN 9971-50-771-4

Printed in Singapore by JBW Printers & Binders Pte. Ltd.

PREFACE

Since the Mid-Seventies Bielefeld University has become an international forum for the exchange of ideas and for the collaboration of scientists and students in Mathematics + Physics from all over the world.

Early on, Bielefeld's Center for Interdisciplinary Research has provided the framework for many such Encounters and Research Years. More recently, activities have concentrated in the research center BiBoS (Bielefeld-Bochum-Stochastik) which began its work in 1984 under an initial grant of the Volkswagen Foundation and now continues its work under a grant from DFG and with support from various other sources. BiBoS has the two intimately related goals of scientific research and exchange of information: it stimulates and hosts research in related fields of mathematics and physics as exemplified by hundreds of BiBoS papers which have found their way into the scientific literature, and it offers an opportunity to a growing number of mathematicians and physicists from all parts of the world to get into touch with recent developments.

As a consequence BiBoS organizes not only technical seminars on recent results but also expository lectures and courses. Some of these are collected here.

Focal points of the present volume are differential equations such as for hydrodynamics, solitary waves, relativistic field theory, stochastic analysis, as well as their interplay which has been attracting a growing interest in recent years. Much remains to be done here and we hope that the book may be of some use to those who want to share the pleasure of observing or participating in the progress of a fascinating field of research.

Thanks are due to all the authors, to the Volkswagen Foundation for its support, and last not least to Ms Litschewsky for her patience and care in assembling the manuscript.

L. Streit, BiBoS
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1988

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INCORRECTNESS AND REGULARIZATION OF EQUATIONS OF HYDRODYNAMICS AND MAGNETOHYDRODYNAMICS WITHOUT DISSIPATION AND TURBULENCE

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As it is well known in gas dynamics the vortex waves, namely, the waves which carry along the gas particles, possess specific characteristic properties. These waves are closely connected with such physical phenomenon as the turbulence. On the other hand, it turns out, that just these waves are connected with an important mathematical phenomenon, namely, with incorrectness of the ideal gas equations, which can be treated as rapid "shaking up" of small initial perturbations. It is clear, that these phenomena are mutually connected. In this book the incorrectness of the ideal gas equations is investigated in detail, and some questions concerning regularization of this incorrect problem are considered. The distinction of properties of this incorrect problems is analyed.

The author wishes to express deep gratitude to Knyaseva I.V., Tsupin V.A. and Omel'yanov G.A., who wrote the text of his lectures on this topic and helped in redaction.

Chapter I.
The Notion of Resonance Incorrectness and the Statement of Main Results

First of all we introduce the notion of strong and complete incorrectness of a self-adjoint operator L in the space L_2. We notes, that in this case the incorrectness of the problem:

$$Lu = f \tag{1.1}$$

is equivalent to the existence of resonances. Really, if f belongs to the kernel of the operator L, i.e. to the subspace of eigenfunctions of the zero eigenvalue, then no solution of problem (1.1) exists. This fact means, that we have a resonance, namely, a small right-hand side, which belongs to this subspce, "swings" the solution, and it becomes infinitely large. The greater the dimension of the kernel is, the simpler a resonance begins.

If the zero point itself is not an eigenvalue of the operator L, but it is a limit point of the spectrum, then the solution of problem (1.1) exists, however, the problem is nevertheless incorrect. We have a resonance for the right-hand sides, which belong to eigen subspaces with eigenvalues close to zero. In addition to that we note, that the solutions large with respect to the norm are associated with small right-hand sides. If the solutions large with respect to the norm L_2 or C are associated with the right-hand sides small with respect to the norm of any Sobolev space W_2^s, then we shall speak, that the incorrectness in strong (M.M. Lavrent'ev, V.I. Ivanov [2,3]). It means, that the values of the operator tend to zero rapidly enough. Namely, the incorrectness is strong, if the product of an eigenvalue by any derivative of the corresponding normed eigenfunction tends to zero.

Moreover, it is important to know, now large the power of the space of functions, which give such a resonance, is. For example, if an arbitrary small neighbourhood of the zero point contains all the eigenvalues except a finite number of them, we can say, that the incorrectness is complete. In other words, there does not exist any infinite-dimensional subspace in L_2, for which the problem were correct. There might be some other situations, where the incorrectness is "sufficiently complete".

EXAMPLE 1. We consider the heat equation:

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2}, \quad x \in [0,\pi],\ t \geq 0, \tag{1.2}$$

with boundary and initial conditions

$$u(0,t) = u(\pi,t) = 0, \quad u(x,0) = u_0(x).$$

The inverse problem is: by knowing the solution at time $t = T$ to restore the initial condition. The solution of problem (1.2) can be repesented in the form:

$$\int_0^{\pi} G(x,\xi,T)u_0(\xi)d\xi = u(x,T), \tag{1.3}$$

where $G(x,\xi,t)$ is the Green function of equation (1.2):

$$G(x,\xi,t) = \frac{2}{\pi}\sum_{n=1}^{\infty} \exp(-n^2 t)\sin(nx)\sin(n\xi).$$

Thus, for the inverse problem we have the Fredholm equation (1.3) of the first kind. The integral operator in the left-hand side of this equation satisfies the conditions of strong and complete incorrectness. Really, the product of the eigenvalue $\lambda_n = \exp(-n^2T)$ be any derivative of the corresponding normed eigenfunction $u_n = \sin(nx)$ tends to zero as $n \to \infty$. Any neighbourhood of the zero point contains all the eighenvalues λ_n, except a finite number of them.

Now we give the accurate definition of strong and complete incorrectness, which concerns nonlinear problems too.

Let $u, f \in L_2(\Omega)$, $\Omega \in R^m$, L be some, generally speaking, nonlinear operator in $L_2(\Omega)$. We consider the problem:

$$L(u) = f \tag{1.4}$$

The norm of the element $g \in L_2(\Omega)$ we shall denote by $\|g\|$. If the sequence of elements $\varphi_n \in L_2(\Omega)$ converges weakly to φ, then we shall write:

$$\varphi_n \xrightarrow[w]{} \varphi.$$

DEFINITION 1. Problem (1.4) will be called strong incorrect, if the inverse operator is not weakly continuous, and if for any integer s there exist such two sequences of functions from $L_2(\Omega)$: $\{u_n^1\}$ and $\{u_n^2\}$, $v \in L_2(\Omega)$ and the numbers $N > 0$, $\delta > 0$, that:

1) $L(u_n^1) - L(u_n^2)$ tends to zero as $n \to \infty$ with respect to the norm of the Sobolev space $W_2^s(\Omega)$;

2) $u_n^k \xrightarrow[n\to\infty]{w} v$, $k = 1,2$ and

   $\|u_n^1 - u_n^2\| \geq \delta$ for $n \geq N$.

This definition is the linear case corresponds to the previous.

DEFINITION 2. Problem (1.4) is called completely incorrect, if there exists such a family of sequence of functions from $L_2(\Omega)$, $\{u_n^k\}$, $k = 1,2,\ldots$, $n = 1,2,\ldots$, and such $N^k > 0$, $\delta^k > 0$ and the functions v, φ^k, f belonging to $L_2(\Omega)$, that for any k the following conditions hold:

1) The condition of incorrectness of this family of sequences:

$$\|L(u_n^k) - L(u_n^1)\| \underset{n\to\infty}{\to} 0,$$

$$u_n^k \xrightarrow[n\to\infty]{w} v$$

and

$$\|u_n^k - u_n^1\| \geq \delta^k \quad \text{for} \quad n \geq N^k, \quad k \neq 1.$$

2) The condition of completeness of this family of sequences:

$$u_n^k \cdot u_n^r \in L_2(\Omega), \quad r = 1,2,\ldots,$$

and

$$(u_n^k - u_n^1)(u_n^2 - u_n^1) \underset{n\to\infty}{\xrightarrow[w]{}} \varphi^k$$

where the set of function $\{\varphi^k, k=2,3,4,\ldots\}$ is dense in a strip $M(A,B) = \{f : f \in L_2(\Omega),\ A \le f \le B\}$ for some $A, B,\ A < B$.

3) The condition of completeness of a family of right-hand sides:

$L(u_n^k) \underset{n\to\infty}{\xrightarrow[w]{}} f$ where the set of functions $\{f\}$ is dense in $L_2(\Omega)$ and either a) $\|L(u_n^k) - f\| \underset{n\to\infty}{\to} 0$ or b) for any $p = 1,2,\ldots.$ $(L(u_n^k))^{2p} \xrightarrow[w]{} f_p$ where the set of vector-functions $(f_1,\ldots,f_p)$ is dense in a strip:

$$M(A,B),\ A = (A_1,\ldots,A_p),\ B = (B_1,\ldots,B_p),\ A_i < B_i,\ i = \overline{1,p},$$

$$M(A,B) = \{(f_1,\ldots,f_p):\ f_i \in L_2(\Omega),\ A_i \le f_i \le B_i\}.$$

REMARK. The condition of completeness of a family of right-hand sides means, that either the right-hand sides $L(u_n^k)$ do not oscillate, or the class of oscillating right-hand sides is wide enough.

EXAMPLE 2. We consider the inverse problem for the Bürgers equation in order to demonstrate the strong and complete incorrectness:

$$\frac{\partial u(x,t)}{\partial t} - \frac{\partial u^2(x,t)}{\partial x} = \frac{\partial^2 u(x,t)}{\partial x^2}$$

$$u(0,t) = u(\pi,t) = 0, \qquad (1.5)$$

$$u(x,0) = u_0(x)$$

Like in Example 1 the inverse problem is: by knowing the solution at time $t = T$ to restore the initial condition. We write the solution (1.5) in the form:

$$L(u_0) = \frac{\int_0^{\pi} G(x,\xi,T)\frac{\partial}{\partial\xi}H(u_0(\xi))d\xi}{\int_0^{\pi} G(x,\xi,T)H(u_0(\xi))d\xi} = u(x,T)$$

where $G(x,\xi,t)$ is the Green function for the heat equation (1.2).

$$H(u_0(\xi)) = \exp(\int_0^{\xi} u_0(s)ds)$$

We consider the set of sequences of functions:

$$u_n^k(x) = \frac{\sin(n+k)x + \frac{\partial\psi(x)}{\partial x}}{2 - \frac{1}{n+k}\cos(n+k)x + \psi(x)} \tag{1.6}$$

$\psi(x) \geq 0$, such that the set

$$\left\{ \frac{\int_0^{\pi} G(x,\xi,T)\frac{\partial\psi(\xi)}{\partial\xi}d\xi}{2 + \int_0^{\pi} G(x,\xi,T)\psi(\xi)d\xi} \right\}$$

is dense in $L_2[0,\pi]$. We have

$$L(u_n^k(x)) = \frac{e^{-(n+k)^2T}\sin(k+n) + \int_0^{\pi} G(x,\xi,T)\frac{\partial\psi(\xi)}{\partial\xi}d\xi}{2 - \frac{1}{(n+k)}e^{-(n+k)^2T}\cos(n+k) + \int_0^{\pi} G(x,\xi,T)\psi(\xi)d\xi} \tag{1.7}$$

It is easy to see

$$\|L(u_n^k) - L(u_n^o)\|_{W_2^s} \underset{n\to\infty}{\to} 0$$

$$u_n^k \xrightarrow[n\to\infty]{w} \frac{\frac{\partial\psi(x)}{\partial x}}{2+\psi(x)},$$

and $\|u_n^k - u_n^o\| \geq \delta^k > 0$ under appropriate choice of δ^k.
Thus, the inverse problem for the Bürgers equation is strong incorrect. By using formula (1.7), we obtain that

$$\| L(u_n^k(x)) - \frac{\int_0^\pi G(x,\xi,T)\frac{\partial\psi(\xi)}{\partial\xi}d\xi}{2+\int_0^\pi G(x,\xi,T)\psi(\xi)d\xi} \| \underset{n\to\infty}{\to} 0$$

i.e. the condition of completeness of right-hand sides is satisfied. So we have to verify only the condition of completeness of a family of a sequence u_n^k. For this purpose it is sufficient to substitute in formula (1.6) $\sin(n+k)x$, $k \neq 0,1$ by the linear combination $\sum_r c_r^k \sin(n+r)x$, and $\cos(n+k)x$ by $\sum_r c_r^k \cos(n+r)x$ respectively, where the coefficients c_r^k were chosen in such a way, that the set $\{\sum_r c_r^k \cos rx,\ k = 1,2,\ldots\}$ is dense in $L_2[0,\pi]$. In this case the conditions of strong incorrectness and of completeness of right-hand sides hold for the new set of sequence u_n^k. In the proof of these statements $\sin(n+k)x$ and $\cos(n+k)x$ are changed by linear combinations of $\sin(n+r)x$ and $\cos(n+r)x$ with the coefficients c_r^k only. Let

$$u_n^1 = 2u_n^o = \frac{2(\sin(nx) + \frac{\partial\psi(x)}{\partial x})}{2 - \frac{\cos(nx)}{n} + \psi(x)},$$

then

$$(u_o^k - u_n^o)(u_n^1 - u_n^o) \xrightarrow[n\to\infty]{w} \frac{\sum_r c_r^k \cos rx - 1}{2(2+\psi(x))^2}$$

This limit is a dense set in $L_2[0,\pi]$ for $k = 1,2,\ldots$. The strong and complete incorrectness of the inverse problem for the Bürgers equation is proved.

As it was already noted, the complete incorrectness characterizes the range of the class of oscillations, which arise under the operator inversion, and the range of the class of right-hand sides, for which the equation is incorrect. In particular, for a linear operator the sufficient condition of such a completeness is the following: the spectrum should have only one point of accumulation, and this point should be the zero point. We consider an example, when the operator spectrum does not possess this property, in this case we have no complete incorrectness.

EXAMPLE 3. Let L be a linear operator on a space of functions from $L_2[-\pi,\pi]$ of the form:

$$Lu(x) = \frac{u(x) - u(-x)}{2} + \int_{-\pi}^{\pi} G(x,\xi,T)\frac{u(\xi) - u(-\xi)}{2}\, d\xi$$

where $G(x,\xi,t)$ is the Green function for the heat equation on the interval $x \in [-\pi,\pi]$. The spectrum of this operator consists of a set of eigenvalues $\lambda_n = \exp(-n^2T)$ with the corresponding eigenfunctions $u_{2n}(x) = \sin nx$, $n = 1,2,\ldots$ and of the point $\lambda = 1$ with an infinite-dimensional eigen subspace of even functions on the interval $[-\pi,\pi]$. It is easy to verify the strong incorrectness, if we take the eigenfunctions $u_n^k(x)$, $k = 1,2,\ldots$, $n = 1,\ldots$ as the sequences $u_n^k(x) = \sin(n+k)x$. Then $L(u_n^k)$ tends to zero as $n \to \infty$ with respect to the norm of the space $W_2^s[-\pi,\pi]$, but the solutions themselves converge to zero only weakly. Nevertheless, the condition of completeness for the set of sequences, which give the incorrectness, cannot be satisfied. Really, let the first condition in Definition 2 hold, then for any k and we have:

$$\| (u_n^k(x) + u_n^k(-x)) - (u_n^1(x) + u_n^1(-x)) \| \underset{n\to\infty}{\to} 0$$

Hence, the product of differences $(u_n^k - u_n^1)(u_n^2 - u_n^1)$ can converge weakly only to even functions, which do not form any dense set in any strip.

Now we introduce the notion of incorrectness for an evolutional system of equations. Actually, an evolutional problem was considered in Examples 1 and 2. It was reduced to the study of incorrectness for resolving operator of the corresponding Cauchy problem.

However, it is convenient to give the definition of incorrectness for an evolutional problem directly. In this case we combine the notions of strong and complete incorrectness, and moreover, we demand, that the incorrectness should exist not only for a fixed time t, but for $t = t_n \underset{n\to\infty}{\to} 0$. The fact is, that the incorrectness of an evolutional problem is also characterized by the time of "shaking up" the small perturbations till the values of the unity order.

Such incorrectness will be called resonance incorrectness. We shall define it for the case of N vectors at one, though the definition will be rather complicated.

We denote by $(L_2(\Omega))^N$ a space of N-dimensional vector-functions, the components of which belong to $L_2(\Omega)$. The norm of the vector $f \in (L_2(\Omega))^N$ is defined by the formula: $\|f\| = \sqrt{\int_\Omega \sum_{i=1}^N f_i^2(x)dx}$ where f_i are the components of the vector f. By analogy, we denote by $(W_2^s(\Omega))^N$ a space of N-dimensional vector-functions with the components, which belongs to the Sobolev space $W_2^s(\Omega)$, $s = 1,2,\ldots$

The norm of the element $f \in (W_2^s(\Omega))^N$ is equal to $\sqrt{\sum_{i=1}^N \|f_i\|^2_{W(\Omega)}}$

We shall use the notations $\langle g,\varphi\rangle = \sum_{i=1}^N g_i\varphi_i$, and $\langle\langle g\rangle\rangle = \sqrt{\sum_{i=1}^N g_i^2}$ where $g, \varphi \in (L_2(\Omega))^N$, g_i, φ_i are the components of vectors g and φ.

Now we define the resonance incorrectness for an arbitrary (both linear and nonlinear) evolutional system

$$\frac{\partial u}{\partial t} + F(u) = 0, \quad u = (u_1,\ldots,u_N) \tag{1.8}$$

where $F(u)$ is a nonlinear operator, $F \in (L_2(\Omega))^N \to L_2(\Omega))^N$. We denote the operator $(\frac{\partial}{\partial t} + F)$ by L. In this case the incorrectness should be given by a rather wide class of initial data.

DEFINITION 3. The evolutional system (1.8) is called resonantly incorrect, if for any s and q there exist such a family of sequences $u_n^k(x,t) \in (L_2(\Omega))^N \cap C^\infty(R_+)$, $k = 1,2,\ldots$, $n = 1,2,3,\ldots$, and such a family of sequences $t_n \to 0$ as $h \to \infty$, that for any k the following conditions hold:

1) The condition of strong incorrectness:

$$\|u_n^k(x,0) - u_n^1(x,0)\|_{(W_2^s(\Omega))^N} \to 0,$$

$$\| L(u_n^k(x,t_n)\|_{(W_2^s(\Omega))^N} \underset{n\to\infty}{\to} 0,$$

$u_n^k(x,t_n) \xrightarrow[n\to\infty]{w} v(x)$ and there exist such $N^k > 0$ and $\delta^k > 0$ that

$$\delta^k \le \|u_n^k(x,t_n) - u_n^1(x,t_n)\|$$

for $n \ge N^k$, $k \ne 1$,

$$t_n = 0(\|u_n^k(x,0)\|^{1/q})$$

2) The condition of completeness of a family of sequences, which give the incorrectness:

$$\langle u_n^k(x,t_n), u_n^r(x,t_n)\rangle \in L_2(\Omega), \quad r = 1,2,\ldots$$

and

$$\langle (u_n^k(x,t_n) - u_n^1(x,t_n)), (u_n^2(x,t_n) - u_n^1(x,t_n))\rangle \xrightarrow[n\to\infty]{w} \varphi^k(x)$$

where the set of functions $\{\varphi^k, k = 1,2,\dots\}$ is dense is a strip $M(A,B)$.

3) The condition of completeness of a class of initial data:

$u_n^k(x,0) \underset{n\to\infty}{\overset{w}{\longrightarrow}} f(x)$ where the set of vector-functions $\{f(x)\}$ is dense in $(L_2(\Omega))^N$. Moreover, either

a) $\|u_n^k(x,0) - f(x)\|_{(L_2(\Omega))^N} \underset{n\to\infty}{\to} 0$

or b) only some components of the vector $u_n^k(x,0)$ converge strongly, and the other components $u_{n_i}^k(x,0)$, $i = (i_1,\dots,i_r)$, $r \le N$, oscillate in a rather arbitrary way, i.e.

$$(u_{n_i}^k(x,0) - f_i(x))^2 \underset{n\to\infty}{\overset{w}{\longrightarrow}} f_i^1(x)$$

$i = (i_1,\dots,i_r)$ where the set of vector-functions

$$\{f^1(x) = (f_{i_1}^1(x),\dots, f_{i_r}^1(x))\}$$

is dense in $(L_2(\Omega))^r$. And for any $p = 2,3,\dots$ we have

$\langle\langle u_n^k - f\rangle\rangle^{2p} \underset{n\to\infty}{\overset{w}{\longrightarrow}} f^p$ where the set of vector-functions $\{f^p=(f^2,f^3,\dots, f^p)\}$ is dense in a strip $M(A,B) \subseteq (L_2(\Omega))^p$, $A = (A_1,\dots,A_p)$, $B = (B_1,\dots,B_p)$.

The number:

$$\gamma_{incorr} = \lim_{n\to\infty} \frac{\ln t_n}{\ln\| u_n^k(x,t_n)\|_{(L_2(\Omega))^N} - \ln\| u_n^k(x,t_n)\|_{(W_2^1(\Omega))^N}}$$

will be called the degree of resonance incorrectness.

EXAMPLE 4. We again consider the problem of inverse heat conduction:

$$\frac{\partial u(x,t)}{\partial t} = -\frac{\partial^2 u(x,t)}{\partial x^2}, \quad x \in [0,\pi], \ t \geq 0,$$
$$u(0,t) = u(\pi,t) = 0 \tag{1.9}$$

Let the initial data have the form:

$$u_n^2(x,0) = 2u_n^1(x,0) = \frac{\sin nx}{n^{s+1}},$$
$$u_n^k(x,0) = \sum_{r=0}^{\infty} c_r^k \frac{\sin(n+r)x}{(n)^{s+1}}, \quad k = 3,4,\ldots \tag{1.10}$$

where the functions $\{\sum_{r=0}^{\infty} c_r^k \sin rx, \ k = 3,4,\ldots\}$ form in $L_2[0,\pi]$ a dense set, and only a finite number of coefficients among c_r^k, $r=1,2,\ldots$ do not vanish. Evidently,

$$||u_n^k(x,0)||_{W_2^s[0,\pi]} \underset{n\to\infty}{\to} 0$$

The exact solution of problem (1.9), (1.10) has the form:

$$u_n^2(x,t) = 2u_n^1(x,t) = 2e^{n^2 t} \frac{\sin(nx)}{n^{s+1}}$$

$$u_n^k(x,t) = \sum_{r=0}^{\infty} c_r^k e^{(n+r)^2 t} \frac{\sin(n+r)x}{n^{s+1}}$$

$$k = 3,4,\ldots$$

We set $t_n = (s+1)\frac{\ln n}{n^2}$, then

$$u_n^k(x,t_n) = \sum_{r=0}^{\infty} c_r^k n^{\frac{(s+1)(n+r)^2}{n^2}} \frac{\sin(n+r)x}{n^{s+1}} =$$