London Mathematical Society
Lecture Note Series 370

Theory of *p*-adic Distributions

Linear and Nonlinear Models

S. Albeverio, A. Yu. Khrennikov and V. M. Shelkovich

CAMBRIDGE

The London Mathematical Society

Theory of *p*-adic Distributions: Linear and Nonlinear Models

S. ALBEVERIO

Institut für Angewandte Mathematik, Universität Bonn,
Endenicherallee 60, D-53115 Bonn, Germany;
H. C. M., I.Z.K.S., SFB 611; BiBoS.

A. YU. KHRENNIKOV

International Center for Mathematical Modeling in Physics
and Cognitive Sciences MSI, Växjö University,
SE-351 95, Växjö, Sweden.

V. M. SHELKOVICH

Department of Mathematics, St. Petersburg State Architecture and
Civil Engineering University, 2-ya Krasnoarmeiskaya 4,
190005, St. Petersburg, Russia.

CAMBRIDGE
UNIVERSITY PRESS

CAMBRIDGE UNIVERSITY PRESS
Cambridge, New York, Melbourne, Madrid, Cape Town, Singapore,
São Paulo, Delhi, Dubai, Tokyo

Cambridge University Press
The Edinburgh Building, Cambridge CB2 8RU, UK

Published in the United States of America by Cambridge University Press, New York

www.cambridge.org
Information on this title: www.cambridge.org/9780521148566

First published 2010

Printed in the United Kingdom at the University Press, Cambridge

A catalogue record for this publication is available from the British Library

ISBN 978-0-521-14856-6 Paperback

293 Second order partial differential equations in Hilbert spaces, G. DA PRATO & J. ZABCZYK
294 Introduction to operator space theory, G. PISIER
295 Geometry and integrability, L. MASON & Y. NUTKU (eds)
296 Lectures on invariant theory, I. DOLGACHEV
297 The homotopy category of simply connected 4-manifolds, H.-J. BAUES
298 Higher operands, higher categories, T. LEINSTER
299 Kleinian groups and hyperbolic 3-manifolds, Y. KOMORI, V. MARKOVIC & C. SERIES (eds)
300 Introduction to Möbius differential geometry, U. HERTRICH-JEROMIN
301 Stable modules and the D(2)-problem, F.E.A. JOHNSON
302 Discrete and continuous nonlinear Schrödinger systems, M.J. ABLOWITZ, B. PRINARI & A.D. TRUBATCH
303 Number theory and algebraic geometry, M. REID & A. SKOROBOGATOV (eds)
304 Groups St Andrews 2001 in Oxford Vol. 1, C.M. CAMPBELL, E.F. ROBERTSON & G.C. SMITH (eds)
305 Groups St Andrews 2001 in Oxford Vol. 2, C.M. CAMPBELL, E.F. ROBERTSON & G.C. SMITH (eds)
306 Peyresq lectures on geometric mechanics and symmetry, J. MONTALDI & T. RATIU (eds)
307 Surveys in combinatorics 2003, C.D. WENSLEY (ed.)
308 Topology, geometry and quantum field theory, U.L. TILLMANN (ed.)
309 Corings and comodules, T. BRZEZINSKI & R. WISBAUER
310 Topics in dynamics and ergodic theory, S. BEZUGLYI & S. KOLYADA (eds)
311 Groups: topological, combinatorial and arithmetic aspects, T. W. MÜLLER (ed.)
312 Foundations of computational mathematics, Minneapolis 2002, FELIPE CUCKER *et al* (eds)
313 Transcendental aspects of algebraic cycles, S. MÜLLER-STACH & C. PETERS (eds)
314 Spectral generalizations of line graphs, D. CVETKOVIĆ, P. ROWLINSON & S. SIMIĆ
315 Structured ring spectra, A. BAKER & B. RICHTER (eds)
316 Linear logic in computer science, T. EHRHARD *et al* (eds)
317 Advances in elliptic curve cryptography, I.F. BLAKE, G. SEROUSSI & N. SMART
318 Perturbation of the boundary in boundary-value problems of partial differential equations, D. HENRY
319 Double affine Hecke algebras, I. CHEREDNIK
320 L-functions and Galois representations, D. BURNS, K. BUZZARD & J. NEKOVÁŘ (eds)
321 Surveys in modern mathematics, V. PRASOLOV & Y. ILYASHENKO (eds)
322 Recent perspectives in random matrix theory and number theory, F. MEZZADRI & N.C. SNAITH (eds)
323 Poisson geometry, deformation quantisation and group representations, S. GUTT *et al* (eds)
324 Singularities and computer algebra, C. LOSSEN & G. PFISTER (eds)
325 Lectures on the Ricci flow, P. TOPPING
326 Modular representations of finite groups of Lie type, J.E. HUMPHREYS
327 Surveys in combinatorics 2005, B.S. WEBB (ed.)
328 Fundamentals of hyperbolic manifolds, R.D. CANARY, A. MARDEN & D.B.A. EPSTEIN (eds)
329 Spaces of Kleinian groups, Y. MINSKY, M. SAKUMA & C. SERIES (eds)
330 Noncommutative localization in algebra and topology, A. RANICKI (ed.)
331 Foundations of computational mathematics, Santander 2005, L. PARDO, A. PINKUS, E. SULI & M. TODD (eds)
332 Handbook of tilting theory, L. ANGELERI HÜGEL, D. HAPPEL & H. KRAUSE (eds)
333 Synthetic differential geometry (2nd Edition), A. KOCK
334 The Navier–Stokes equations, P. G. DRAZIN & N. RILEY
335 Lectures on the combinatorics of free probability, A. NICA & R. SPEICHER
336 Integral closure of ideals, rings, and modules, I. SWANSON & C. HUNEKE
337 Methods in Banach space theory, J.M.F. CASTILLO & W.B. JOHNSON (eds)
338 Surveys in geometry and number theory, N. YOUNG (ed.)
339 Groups St Andrews 2005 Vol. 1, C.M. CAMPBELL, M.R. QUICK, E.F. ROBERTSON & G.C. SMITH (eds)
340 Groups St Andrews 2005 Vol. 2, C.M. CAMPBELL, M.R. QUICK, E.F. ROBERTSON & G.C. SMITH (eds)
341 Ranks of elliptic curves and random matrix theory, J.B. CONREY, D.W. FARMER, F. MEZZADRI & N.C. SNAITH (eds)
342 Elliptic cohomology, H.R. MILLER & D.C. RAVENEL (eds)
343 Algebraic cycles and motives Vol. 1, J. NAGEL & C. PETERS (eds)
344 Algebraic cycles and motives Vol. 2, J. NAGEL & C. PETERS (eds)
345 Algebraic and analytic geometry, A. NEEMAN
346 Surveys in combinatorics, 2007, A. HILTON & J. TALBOT (eds)
347 Surveys in contemporary mathematics, N. YOUNG & Y. CHOI (eds)
348 Transcendental dynamics and complex analysis, P. RIPPON & G. STALLARD (eds)
349 Model theory with applications to algebra and analysis Vol 1, Z. CHATZIDAKIS, D. MACPHERSON, A. PILLAY & A. WILKIE (eds)
350 Model theory with applications to algebra and analysis Vol 2, Z. CHATZIDAKIS, D. MACPHERSON, A. PILLAY & A. WILKIE (eds)
351 Finite von Neumann algebras and masas, A. SINCLAIR & R. SMITH
352 Number theory and polynomials, J. MCKEE & C. SMYTH (eds)
353 Trends in stochastic analysis, J. BLATH, P. MÖRTERS & M. SCHEUTZOW (eds)
354 Groups and analysis, K. TENT (ed)
355 Non-equilibrium statistical mechanics and turbulence, J. CARDY, G. FALKOVICH & K. GAWEDZKI, S. NAZARENKO & O.V. ZABORONSKI (eds)
356 Elliptic curves and big Galois representations, D. DELBOURGO
357 Algebraic theory of differential equations, M.A.H MACCALLUM & A.V. MIKHAILOV (eds)
358 Geometric and cohomological methods in group theory, M. BRIDSON, P. KROPHOLLER & I. LEARY (eds)
359 Moduli spaces and vector bundles, L. BRAMBILA-PAZ, S.B. BRADLOW, O. GARCÍA-PRADA & S. RAMANAN (eds)
360 Zariski geometries, B. ZILBER
361 Words: Notes on verbal width in groups, D. SEGAL
362 Differential tensor algebras and their module categories, R. BAUTISTA, L. SALMERÓN & R. ZUAZUA
363 Foundations of computational mathematics, Hong Kong 2008, M.J. TODD, F. CUCKER & A. PINKUS (eds)
364 Partial differential equations and fluid mechanics, J.C. ROBINSON & J.L. RODRIGO (eds)
365 Surveys in combinatorics 2009, S. HUCZYNSKA, J.D. MITCHELL & C.M. RONEY-DOUGAL (eds)
366 Highly oscillatory problems, B. ENGQUIST, A. FOKAS, E. HAIRER & A. ISERLES (eds)
367 Random matrices: High dimensional phenomena, G. BLOWER
368 Geometry of Riemann surfaces, F. P. GARDINER, G. GONZALEZ-DIEZ & C. KOUROUNIOTIS (eds)
369 Epidemics and rumours in complex networks, M. DRAIEF & L. MASSOULIÉ

To Solvejg, Irina, Evgenia

Preface

For a few hundred years theoretical physics has been developed on the basis of real and, later, complex numbers. This mathematical model of physical reality survived even in the process of the transition from classical to quantum physics – complex numbers became more important than real, but not essentially more so than in the Fourier analysis which was already being used, e.g., in classical electrodynamics and acoustics. However, in the last 20 years the field of p-adic numbers $\mathbb{Q}_p$ (as well as its algebraic extensions, including the field of complex p-adic numbers $\mathbb{C}_p$) has been intensively used in theoretical and mathematical physics (see [1]–[3], [10]–[15], [24], [35], [38], [43], [44], [55], [64], [65], [77]–[79], [89], [94], [115]–[122], [123], [124], [133], [157], [158], [174], [198], [241]–[246] and the references therein). Thus, notwithstanding the fact that p-adic numbers were only discovered by K. Hensel around the end of the nineteenth century, the theory of p-adic numbers has already penetrated intensively into several areas of mathematics and its applications.

The starting point of applications of p-adic numbers to theoretical physics was an attempt to solve one of the most exciting problems of modern physics, namely, to combine consistently quantum mechanics and gravity, thus to create a theory of quantum gravity. In spite of the considerable success of some models, there is still no satisfactory general theory. In the physical literature one could even find discussions that one should not exclude the possibility that either quantum mechanics or gravitation theory is wrong, and radical new ideas are needed to solve the problems of quantum gravity. Physical models with p-adic space appeared as an attempt to provide a less radical approach to such problems (see [39], [40], [41], [77], [79]). The main idea behind p-adic theoretic physics (at least in the first period of its development) was that troubles with consistency of quantum mechanics and gravity are induced by the use of an infinitely divisible real continuum as the basic mathematical model of the underlying physical space-time. One should then look for new mathematical models of space-time and of structures defined on it. The fields of p-adic numbers provide an excellent new possibility. In the literature on applications of p-adic numbers in physics numerous arguments have been put forward in favor of the use of p-adic numbers. For example, in cosmology and string theory, speculations have been formulated according to which space-time at the so-called Planck scale (of the order of 10^{-23} cm) should be disordered and the Archimedean axiom might be violated for measurements which would be performed on such scales (see [115], [116], [245], [246]). Since the fields of p-adic numbers are disordered as well as non-Archimedean it seems appropriate to exploit such fields in this sense. We also mention the following argument [116], [241]. If one starts with the field of rational numbers[1] $\mathbb{Q}$ and wants to obtain a complete field with K valuation ("absolute value"), then, as a consequence of the famous Ostrovsky theorem (Theorem 1.3.2 in Chapter 1), there are only two possibilities for K: either K is the field of real numbers or K is one of the fields of p-adic numbers $\mathbb{Q}_p$.

p-adic based models can be considered as having given an important new contribution to string theory, gravity and cosmology. However, the most important consequence of this p-adic physical activity was that it induced a kind of "Carnot cycle": physics $\to$ mathematics $\to$ physics... Physical applications induce developments of new mathematical theories which in their turn induce new physical applications (which in their turn induce new mathematical theories and so on). p-adic string theory, gravity and cosmology stimulated development, and new applications of p-adic Fourier analysis, the theory of p-adic distributions and pseudo-differential equations, the theory of self-adjoint operators in $\mathcal{L}^2(\mathbb{Q}_p)$, as well as, e.g., the theory of stochastic differential equations over the field of p-adic numbers, Feynman path integration over p-adics, the theory of p-adic valued probabilities and dynamical systems (see, e.g., the monographs [36], [115], [130], [157], [162], [241] and the papers [11], [13], [32]–[34], [129]–[130], [249]–[251]). In their turn these mathematical theories provide new possibilities for physical applications of p-adic mathematics – e.g. in the theory of disordered systems (spin glasses) ([43], [44], [162], [164], [199]). Applications were, however, not only restricted to physics. There were also proposed p-adic models in psychology, cognitive and social sciences, and, e.g., in biology, image analysis (see, e.g., [116], [117]).

[1] We recall that in all real physical experiments measurements have only a finite precision. Therefore the result of any measurement is always a rational number, see [241], [116] for discussions.

Those new applications stimulated the development of new branches of p-adic analysis, in particular, the theory of p-adic wavelets ([21], [131], [132], [142], [143], [144], [146], [147], [148], [162], [163], [164], [165], [225]). The latter theory provides new possibilities for the study of p-adic pseudo-differential equations, surprisingly not only linear but also nonlinear ones ([23], [143], [162], [166]). The wavelet theory also gives a proper technique to solve the p-adic Schrödinger equation with point interactions ([31], [169]). And again the possibility to obtain wavelet-like solutions of the p-adic Schrödinger equation implied interesting physical consequences [133].

Further applications demonstrated that one should consider more general non-Archimedean models (e.g., for spin glasses, psychology, neurophysiology) and develop a corresponding analysis on general ultrametric spaces ([131], [132], [135]–[138], [162]).

Of course, one should never forget that mathematics has its own self-stimulating forces in the development of new concepts and formalisms. Concerning p-adic analysis we should mention the development of harmonic analysis on general locally compact groups as well as its applications inside mathematics ([203], [105]). However, many problems are intrinsically of a p-adic nature and they could not be formulated or even if formulated they could not be studied in such details without the p-adic framework. We remark that a part of p-adic analysis was also developed intensively without reference to physics; see, e.g., Gouvêa [98], Katok [112], Koblitz [152], Robert [204], Schikhof [214], Taibleson [230]. Nevertheless, many fundamental developments in p-adic analysis, such as [115]–[118], [157], [241], were indeed stimulated by physics and other applications.

When one speaks about p-adic analysis and applications, one should always mention which kind of p-adic model is under consideration. There are two main frameworks: either concerned with the relevant maps going from $\mathbb{Q}_p$ into $\mathbb{C}$, where $\mathbb{C}$ is the field of complex numbers, or from $\mathbb{Q}_p$ into $\mathbb{Q}_p$. The first stage of development of p-adic mathematical physics for the model with $\mathbb{C}$-valued maps as well as the corresponding analysis was presented in the excellent book by Vladimirov, Volovich, and Zelenov [241] which became like a kind of "Bible" for physicists using p-adic models as well as for mathematicians looking for new problems connected with p-adics. Let us also mention the book by Kochubei [157], which is also basically concerned with applications, especially covering probabilistic aspects. Physical, cognitive and psychological models using $\mathbb{Q}_p$-valued maps as well as the corresponding analysis were presented in a series of monographs by one of the authors of this book [115]–[117]. The latter models are more extensively represented (at the book level) than the models based on maps from $\mathbb{Q}_p$ into $\mathbb{C}$.

We think the time has come to present the new mathematical theories which were developed in connection with p-adic mathematical physics in a monograph which would, on the one hand, update the book [241] by presenting and using new recent results and, on the other hand, open new possibilities for the development of p-adic analysis for maps from $\mathbb{Q}_p$ into $\mathbb{C}$. Our book should serve for such a purpose. In this book a wide-ranging survey of many important topics in p-adic analysis is presented. Moreover, it also gives a self-contained presentation of p-adic analysis itself.

The main attention is given to the development of p-adic distribution theory, including nonlinear models, Fourier and wavelet analysis, the theory of pseudo-differential operators and equations, Tauberian theorems, and asymptotic methods.

The book consists of fourteen chapters and four appendixes. Every chapter begins with a brief summary of its contents and a brief survey of related results. Chapters 1–5 cover standard material to introduce the reader to the theory of p-adic numbers, p-adic functions and p-adic distributions. These chapters are based first of all on the books by Vladimirov, Vololvich, Zelenov [241], Taibleson [230], and Katok [112], as well as on those by Koblitz [152], Gouvêa [98], Robert [204], and Schikhof [214]. The next nine chapters are more advanced and are based on the authors' original results (see References). These chapters represent new developments in p-adic harmonic analysis, in the p-adic theory of generalized functions, in the p-adic theory of wavelets, and in p-adic pseudo-differential operators and equations. In each chapter we compare p-adic results with the corresponding results in the real case.

In Chapter 6, we develop the theory of p-adic *associated homogeneous distributions* and *quasi associated homogeneous distributions*. The results of this chapter are closely connected with the theory of real *quasi associated homogeneous distributions*, which is presented in Appendix A and gives the solution of an important problem. In Chapter 7, we introduce the p-adic Lizorkin spaces of test functions and distributions and study their properties. These spaces are invariant under pseudo-differential operators (in particular, fractional operators), and, consequently, they provide "natural" domains of definition for them (see Chapters 9, 10). We would like to recall that in the real case this type of space was introduced by P. I. Lizorkin [180]–[182]. In the real setting such spaces play a key role in some problems related to fractional operators and are used in applications [210], [211], [206]. In Chapter 8, we develop a p-adic *wavelet theory*, as well as a p-adic *multiresolution analysis* and construct wavelet bases. The wavelet theory plays a key role in applications of p-adic analysis and gives a new powerful technique for solving p-adic problems (see Chapters 9–11). In Chapter 9 one class of multidimensional pseudo-differential operators is studied. This class includes fractional operators. In this chapter the spectral theory of pseudo-differential operators is developed. We derive a criterion for multidimensional p-adic pseudo-differential operators to have multidimensional wavelets (constructed in Chapter 8) and characteristic functions of balls as eigenfunctions. In particular, the multidimensional wavelets are eigenfunctions of the Taibleson fractional operator. In Chapter 10, linear and nonlinear p-adic evolutionary pseudo-differential equations are studied. To solve the Cauchy problems we develop the *"variable separation method"* which is an analog of the classical Fourier method. This method is based on the fact that the wavelets constructed in Chapter 8 are eigenfunctions of pseudo-differential operators constructed in Chapter 9. In Chapter 11 we continue the investigation of p-adic Schrödinger-type operators with point interactions started by A. N. Kochubei [156], and study $D^{\alpha} + V_Y$, where D^{α} $(\alpha > 0)$ is the operator of fractional differentiation (which was studied in Chapter 9) and V_Y is a singular potential containing the Dirac delta functions. In Chapter 12, we extend the notion of regular variation introduced by J. Karamata to the p-adic case and prove multidimensional *Tauberian type theorems* for distributions. In Chapter 13, we study the asymptotic behavior of p-adic singular Fourier integrals. All constructed asymptotic formulas have the *stabilization* property (in contrast to the asymptotic formulas in the real setting). Theorems which give asymptotic expansions of singular Fourier integrals are related with the Abelian type theorems which were proved in Chapter 12. In Chapter 14, we develop the nonlinear theory of distributions (generalized functions). We construct the algebraic technique which allows us to solve both linear and nonlinear singular problems of the p-adic analysis related with the theory of distributions. Notwithstanding the fact that the real case problems related with the multiplication of distributions were studied in many books and papers (see [57], [66], [68], [86], [97], [108], [176]–[179], [194]–[197], [218]–[222]), p-adic analogs of above mentioned problems have not been studied so far.

In our opinion, our book could serve as a basic course on p-adics and its applications, as well as for graduate courses such as: "the theory of p-adic distributions", "p-adic harmonic analysis", "the theory of p-adic wavelets", "the theory of p-adic pseudo-differential operators".

This research was supported in part by the grant of Växjö University: "Mathematical Modelling". The first and the third authors (S. A. and V. S.) were also supported in part by DFG Project 436 RUS 113/809 and DFG Project 436 RUS 113/951. The third author (V. S.) was also supported in part by Grants 05-01-04002-NNIOa and 09-01-00162 of Russian Foundation for Basic Research.

We wish to thank Ya. I. Belopolskaya, R. L. Benedetto, B. G. Dragovich, Yu. N. Drozzinov, A. Escassut, W. Karwowski, A. V. Kosyak, S. V. Kozyrev, S. Kuzhel, Hassan A. Marić, G. Parisi, V. I. Polischook, M. A. Skopina, V. S. Vladimirov, I. V. Volovich, B. I. Zavialov, V. V. Zharinov, K. Yasuda for fruitful discussions. We also are greatly indebted to Evgenia Sorokina for careful reading of this book and elimination of misprints.

Sergio Albeverio, Andrei Khrennikov, Vladimir Shelkovich

Bonn–Växjö–St. Petersburg–Moscow

Contents

1

p-adic numbers

1.1. Introduction

In this chapter some basic facts on the field of p-adic numbers $\mathbb{Q}_p$ are presented. Here we follow some sections from the books [47], [96], [98], [152], [241], and especially from the textbook [112]. Section 1.9.3 follows [254] and [241, 1.6.]. Section 1.9.4 is based on [163] and [162, 2.4.].

1.2. Archimedean and non-Archimedean normed fields

Denote by $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{C}$ the sets of positive integers, rational numbers, integers, real numbers, nonnegative real numbers and complex numbers, respectively, and set $\mathbb{N}_0 = 0 \cup \mathbb{N}$.

Definition 1.1. Let X be a nonempty set. A *distance* or *metric* on X is a map $d : X \times X \to \mathbb{R}_+$ such that for all $x, y, z \in X$ we have

(1) $d(x, y) = 0 \Leftrightarrow x = y$;

(2) $d(x, y) = d(y, x)$;

(3) $d(x, y) \le d(x, z) + d(z, y)$ (*triangle inequality*).

A set together with a metric is called a *metric space*.

A metric d is called *non-Archimedean* (or *ultra-metric*) if it satisfies the additional condition

(3′) $d(x, y) \le \max(d(x, z), d(z, y))$ (*strong triangle inequality*).

The corresponding metric space is called an *ultrametric space*.

Since $\max(d(x, z), d(z, y))$ does not exceed the sum $d(x, z) + d(z, y)$, condition (3′), the *strong triangle inequality*, implies condition (3), the *triangle inequality*.

The same set X can give rise to many different metric spaces (X, d).