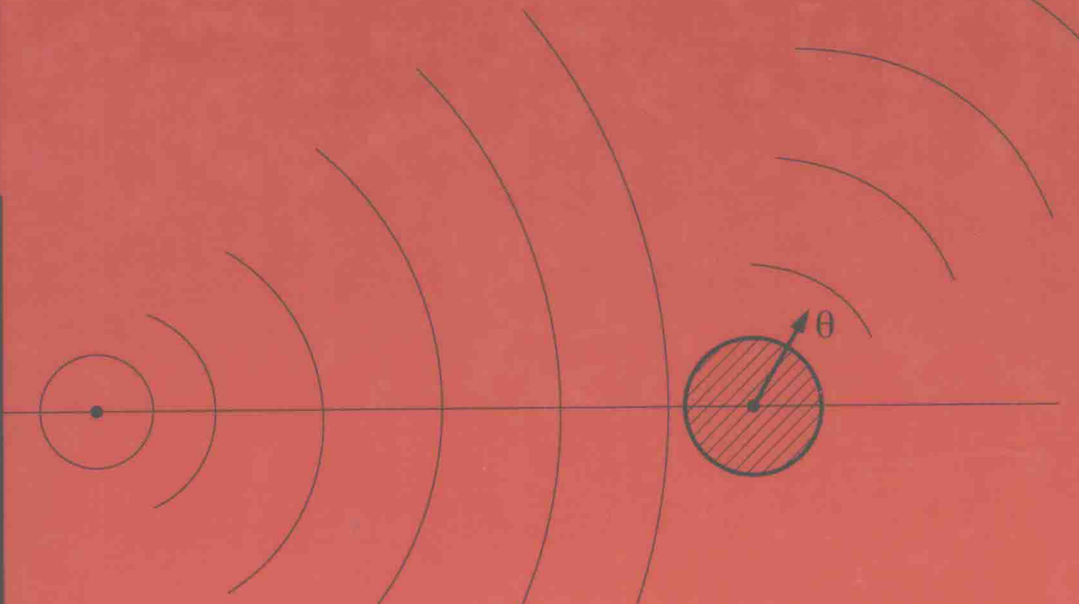
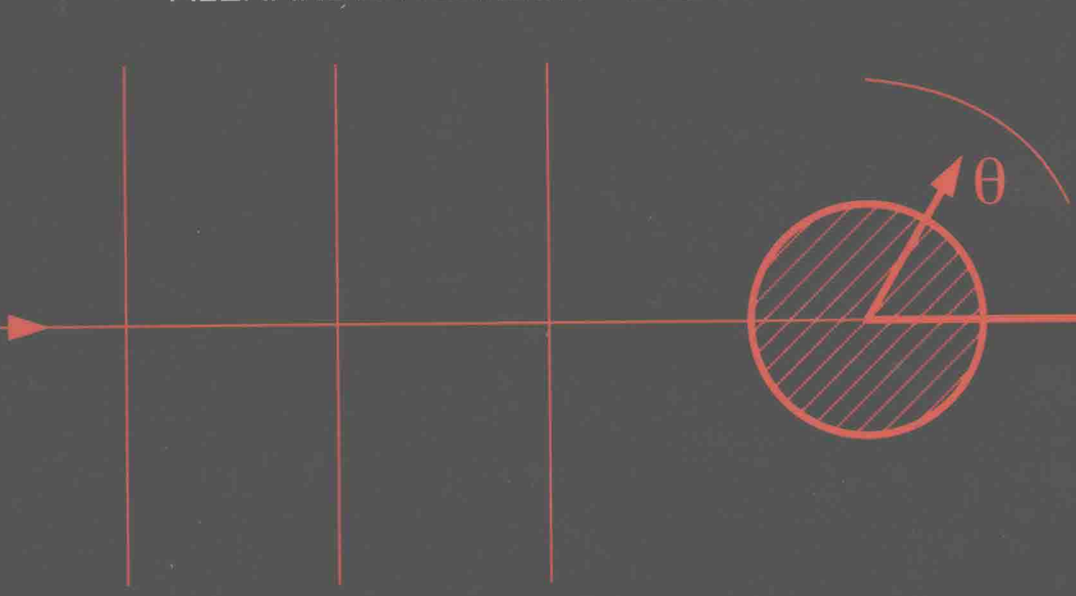


DISPERSION DECAY AND SCATTERING THEORY

ALEXANDER KOMECH • ELENA KOPYLOVA



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A JOHN WILEY & SONS, INC., PUBLICATION

Published by John Wiley & Sons, Inc., Hoboken, New Jersey
Published simultaneously in Canada

Library of Congress Cataloging-in-Publication Data:

Komech, A. I., 1946–
Dispersion decay and scattering theory / Alexander Komech, Elena Kopylova.
p. cm.
Includes bibliographical references and index.
ISBN 978-1-118-34182-7 (cloth)
1. Klein-Gordon equation. 2. Spectral theory (Mathematics) 3. Scattering (Mathematics) I. Kopylova, Elena, 1960- II. Title.
QC174.26.W28K646 2012
530.12'4—dc23 2012007292

Printed in the United States of America.

10 9 8 7 6 5 4 3 2 1

FOREWORD

The book is a concise introduction to the dispersion decay and its applications to the scattering and spectral theory for the Schrödinger, Klein-Gordon, and wave equations. We expose the Agmon, Jensen, and Kato results on analytical properties of the resolvent in weighted Sobolev norms and applications to the spectral and scattering theory. The course is intended for readers who have a nodding acquaintance with the Fourier transform of distributions, the Sobolev embedding theorems, and the Fredholm Theorem.

PREFACE

We present the extended lecture notes of the course delivered by one of the authors in the Faculty of Mathematics of Vienna University in the spring 2009 for graduate students IV–V years.

Our aim is to give an introduction to spectral methods for the Schrödinger and Klein-Gordon equations with applications to a dispersion time-decay and scattering theory. This method relies on analytical properties of the resolvent: high energy decay and low energy asymptotics of the resolvent, and the limiting absorption principle (a smoothness of the resolvent in the continuous spectrum).

This strategy in the dispersion time-decay was introduced by Vainberg for general hyperbolic equations with constant coefficients outside a compact region, and initial functions with compact support. The approach was extended by Agmon, Jensen, Kato, Murata and others to the Schrödinger equation with generic potentials of algebraic decay, and initial functions from the weighted Sobolev spaces. These results play a crucial role in the study of asymptotic stability of solutions to nonlinear Schrödinger equations, see [7, 8, 11, 64, 65, 80, 81].

We present the Agmon, Jensen, and Kato results for the first time in the textbook literature. Then we apply them to a new dynamical justification of the scattering cross section via the *limiting amplitude principle* and convergence of the "spherical limit amplitudes" to the "plane limit amplitudes". We also present an extension of the methods and results to the Klein-Gordon and wave equations obtained in [45, 48, 51]. Recently the results were successfully applied for proving asymptotic stability for the kinks of relativistic invariant nonlinear Ginzburg-Landau equations [43, 44].

The course is intended for readers who have a nodding acquaintance with the Fourier transform of distributions, the Sobolev embedding theorems, and the Fredholm Theorem.

We do not touch alternative approaches to the dispersion decay and scattering (Birman-Kato theory [70], Strichartz estimates [38], Mourre estimates [26], Hunziker-Sigal method of minimal escape velocity [28, 29], and other) not to overburden the exposition.

In Sections 1 and 2 we collect basic concepts and facts which we need: the Fourier transform of distributions, the Sobolev embedding theorems, the Fredholm Theorem, and basic technique of pseudodifferential operators (everything is covered, e.g., by [77] or [40]). In Sections 3–15 we establish basic properties of the Schrödinger equation. In the central sections 16–22 we present the Agmon-Jensen-Kato spectral theory of the dispersion decay in the weighted Sobolev norms. In the remaining sections 23–41 we apply the dispersion decay to scattering and spectral theories, to a justification of scattering cross section, and to a weighted energy decay for 3D Klein-Gordon and wave equations with a potential.

One of the cornerstones of the Agmon-Jensen-Kato approach is the high energy decay of the resolvent in the weighted Sobolev norms, which was stated by Agmon in [1, (A.2')]. We give a complete proof explaining all related details: the Sobolev Trace Theorem, the Hölder continuity of the traces, the Sokhotsky-Plemelj formula, etc. The next cornerstones are Kato's theorem on the absence of embedded eigenvalues and Agmon's theorem on the decay of the eigenfunctions. We give complete streamlined proofs.

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Moscow-Vienna
January, 2012

ACKNOWLEDGMENTS

The authors thank B. R. Vainberg for useful remarks. The authors are indebted to the Faculty of Mathematics of Vienna University and Institute for Information Transmission Problems of RAS for excellent conditions for the work. The book was written under the support of the Alexander von Humboldt Award of A. I. Komech, and by the Austrian Science Fund (FWF): P22198-N13 and M1329-N13.

K. A. I. and K. E. A.

Keywords: dispersion decay, Schrödinger and Klein-Gordon equations, Schrödinger operator, potential, resolvent, Fredholm Theorem, distribution, Fourier transform, Fourier-Laplace transform, pseudodifferential operator, Sobolev norm, Sobolev space, Sobolev Embedding Theorem, weighted spaces, continuous spectrum, convolution, Born series, limiting absorption principle, asymptotic completeness, wave operators, scattering operator, S-matrix, T-operator, limiting amplitude principle, plane limit amplitude, spherical limit amplitude, Wiener condition, scattering cross section, Lippmann-Schwinger equation.

2000 Mathematics Subject Classification: 35L10, 34L25, 47A40, 81U05.

INTRODUCTION

Dispersion decay and scattering The main subject of our book is the study of wave radiation and scattering for solutions to the Schrödinger and Klein-Gordon equations with a decaying potential

$$i\dot{\psi}(x,t) = H\psi(x,t) := -\Delta\psi(x,t) + V(x)\psi(x,t)\ , \quad x \in \mathbb{R}^3\ , \qquad (0.1)$$

$$\ddot{\psi}(x,t) = \Delta\psi(x,t) - m^2\psi(x,t) - V(x)\psi(x,t), \quad x \in \mathbb{R}^3\ , \qquad (0.2)$$

which are the basic wave equations of quantum mechanics, introduced in 1925–1926. The key peculiarity of the wave processes is the energy propagation and energy radiation to infinity known since Huygens' "Treatise on light" (1678).

This radiation is demonstrated by the dispersion time decay which is a fundamental property of solutions to general linear hyperbolic partial differential equations. The decay was first justified by Kirchhoff about 1882 for solutions to the *acoustic equation*, which is the Klein-Gordon equation (0.2) with $m = 0$ and $V(x) = 0$. Namely, Kirchhoff discovered the famous formula (39.7), (39.8) which, in particular, implies the *strong Huygens principle* for the acoustic equation:

$$\psi(x,t) = 0 \quad \text{for} \quad |x| < |t| - R_0 \ \text{and for}\ |x| > |t| + R_0 \qquad (0.3)$$

if

$$\psi(x,0) = 0\ ,\ \dot{\psi}(x,0) = 0 \ \text{for} \ \ |x| > R_0. \qquad (0.4)$$

In particular, (0.3) implies

$$\psi(x,t)=0 \quad \text{for } |x|<R \text{ and } |t|>R_0+R \tag{0.5}$$

for any $R>0$. This *wave divergence* was widely recognized in theoretical physics in the nineteenth and twentieth centuries. In particular, it was one of the key inspirations for Bohr's theory of radiation induced by the quantum transitions.

However, a mathematical justification of this phenomenon was discovered only after 1960 by Lax, Morawetz, Phillips, and Vainberg for wave and Klein-Gordon equations and extended by Ginibre and Velo, Rauch, and others for the Schrödinger equation in the theory of *local energy decay*:

$$\int_{|x|<R} |\psi(x,t)|^2 dx \to 0\,, \qquad |t|\to\infty\,, \tag{0.6}$$

for any $R>0$ under condition of type (0.4) on initial data and a suitable condition on the potential $V(x)$.

In 1979, Jensen and Kato proved a stronger decay in the weighted Sobolev norms for the Schrödinger equation (0.1). In particular, for the free Schrödinger equation with $V(x)=0$,

$$\|\langle x\rangle^{-\sigma}\psi(x,t)\|\to 0\,, \qquad |t|\to\infty\,, \tag{0.7}$$

for a sufficiently large $\sigma>0$ if

$$\|\langle x\rangle^{\sigma}\psi(x,0)\|<\infty\,, \tag{0.8}$$

where $\langle x\rangle=(1+|x|)^{1/2}$ and $\|\cdot\|$ stands for the norm in the Hilbert space $\mathcal{L}^2 := L^2(\mathbb{R}^3)$. Obviously, condition (0.8) is an analog of (0.4), while the decay (0.7) generalizes (0.5) and (0.6). The approach relies on the Agmon analytical theory of the resolvent [1]. Murata extended these methods and results to more general equations of the Schrödinger type [62]. Recently the decay in the weighted Sobolev norms was extended to the wave and Klein-Gordon equations [45]–[51].

The decay (0.7) obviously does not hold for solutions of type $\psi(x)e^{i\omega t}$ to (0.1) with real ω if they exist. In this case $H\psi=\omega\psi$, i.e., $\psi(x)$ is the eigenfunction of H. However, it turns out that the decay holds for solutions with initial states $\psi(x,0)\in X_c$, where X_c is the subspace of functions from $\mathcal{L}^2$ which are orthogonal to all eigenfunctions.

This decay allows to clarify significantly the structure of the trajectories. Namely, the decay implies that the term $V(x)\psi(x,t)$ in (0.1) dies down as $|t|\to\infty$, and hence the equation reduces to the free equation with $V(x)=0$. Respectively, one could expect that $\psi(x,t)$ converges for large times to the corresponding solutions of the free Schrödinger equation:

$$\psi(x,t)\sim\phi_\pm(x,t)\,, \qquad t\to\pm\infty\,. \tag{0.9}$$

Of course, $\phi_+(x,t)=\phi_-(x,t)$ if $V(x)=0$. Therefore, the difference between $\phi_\pm(x,t)$ reflects the properties of the potential $V(x)$. The map $S:\phi_-(\cdot,0)\to\phi_+(\cdot,0)$ is called the *scattering operator*.

The decay (0.7) for $\psi(x,0) \in X_c$ allows to prove *asymptotic completeness* in the scattering, which means that $S : \mathcal{L}^2 \to \mathcal{L}^2$ is a unitary operator. The decay also allows to give a dynamical justification for the *quantum scattering cross section* [42].

Note that the decay in $L^2(\mathbb{R}^3)$ for solutions to the Schrödinger and Klein-Gordon equations does not hold due to the conservation of the corresponding charge Q and energy E: for the Schrödinger equation

$$Q := \int_{\mathbb{R}^3} |\psi(x,t)|^2 dx, \quad E := \int_{\mathbb{R}^3} \overline{\psi(x,t)} H \psi(x,t) dx\,, \tag{0.10}$$

and for the Klein-Gordon equation

$$Q := \operatorname{Im} \int_{\mathbb{R}^3} \overline{\psi(x,t)} \dot{\psi}(x,t) dx,$$
$$E := \int_{\mathbb{R}^3} [|\dot{\psi}(x,t)|^2 + \overline{\psi(x,t)}(H + m^2)\psi(x,t)] dx\,. \tag{0.11}$$

Contents The main goal of the present lectures is to give an introduction to the dispersion decay in weighted norms and its applications. We assume that the potential $V(x)$ is a real-valued continuous function which decays at infinity:

$$V(x) \in C(\mathbb{R}^3, \mathbb{R}), \quad \sup_{x \in \mathbb{R}^3} \langle x \rangle^\beta |V(x)| < \infty\,, \tag{0.12}$$

where $\beta > 0$ is sufficiently large.

In Chapter 1 we recall basic concepts of tempered distribution theory, formulas for the Fourier transform, and functional spaces that we will use. We also calculate an integral representation for the solution to the free Schrödinger equation (0.1) corresponding to $V = 0$.

In Chapter 2 we prove well-posedness of the initial problem for the Schrödinger equation (0.1): for initial data $\psi(0) \in \mathcal{L}^2$, the solution exists and is unique, and the corresponding *dynamical group* $U(t) : \psi(0) \mapsto \psi(t)$ is unitary in $\mathcal{L}^2$. For the proof we apply the contraction mapping principle to the integral Duhamel representation which is equivalent to (0.1). The total charge and energy (0.10) are conserved.

In Chapter 3 we calculate an integral representation for solutions to the *free stationary Schrödinger equation* corresponding to $V(x) = 0$. Further, we prove analyticity and some bounds for the resolvent $R(\omega) := (H - \omega)^{-1}$ of the Schrödinger operator (0.1). The resolvent is analytic for $\omega \in \mathbb{C} \setminus [V_0, \infty)$, where $V_0 = \min_{x \in \mathbb{R}} V(x)$.

In Chapter 4 we establish a spectral representation of type (0.15) for solutions to (0.1) and prove that the resolvent $R(\omega)$ admits the meromorphic continuation to $\omega \in [V_0, 0)$ with the poles at the discrete set of points $\omega_j \in [V_0, 0)$ which are eigenvalues of H with the corresponding eigenfunctions $\psi_j \in \mathcal{L}^2$:

$$H\psi_j = \omega_j \psi_j\,. \tag{0.13}$$

The subspace X_d of the discrete spectrum, generated by the eigenfunctions, is finite dimensional for *generic potentials* V. In conclusion we prove the famous Kato Theorem on the absence of the positive *embedded eigenvalues*.

In Chapters 5–7 we establish the asymptotic behavior of the resolvent $R(\omega)$ for small and large ω [see (0.28) and (0.29) below] and establish the *limiting absorption principle*

$$R(r \pm i\varepsilon) \to R(r \pm i0)\,, \quad \varepsilon \to 0+\,, \quad r \geq 0\,, \tag{0.14}$$

in an appropriate operator norm. We assume *spectral condition* (19.9), which means that the *point* $\lambda = 0$ *is neither an eigenvalue nor a resonance* for the Schrödinger operator H. The condition holds for *generic potentials*. These properties allow to justify the *spectral representation* for solutions to (0.1),

$$\psi(t) = \sum_1^N C_j \psi_j e^{-i\omega_j t} + \frac{1}{2\pi i}\int_0^\infty e^{-i\omega t}\Big[R(\omega+i0) - R(\omega-i0)\Big]\psi(0)\,d\omega\,. \tag{0.15}$$

The last integral represents the solutions $\psi(x,t)$ with initial states $\psi(x,0) \in X_c$, where X_c is the space of the continuous spectrum of H. For these solutions we prove the dispersion decay

$$\|\langle x\rangle^{-\sigma}\psi(x,t)\| \leq C(\sigma)\langle t\rangle^{-3/2}\|\langle x\rangle^{\sigma}\psi(x,0)\|\,, \quad \sigma > 5/2\,, \tag{0.16}$$

established by Jensen and Kato [35].

In Chapter 8 we deduce (0.9) as a corollary of (0.16). More precisely, for $\psi(x,0) \in X_c$

$$\psi(x,t) = \psi_\pm(x,t) + r_\pm(x,t)\,, \tag{0.17}$$

where $\psi_\pm(x,t)$ are the corresponding solutions to the free Schrödinger equation and the remainder decays in the $\mathcal{L}^2$-norm:

$$\|r_\pm(\cdot,t)\| \to 0, \qquad t \to \pm\infty\,. \tag{0.18}$$

Each *wave operator*

$$W_\pm : \psi(x,0) \mapsto \psi_\pm(x,0) \tag{0.19}$$

is an isometry of X_c onto $\mathcal{L}^2$, so the *scattering operator*

$$S = W_+ W_-^{-1} : \psi_-(x,0) \to \psi_+(x,0) \tag{0.20}$$

is unitary in $\mathcal{L}^2$ (see Fig. 1). We apply the wave operators for the spectral resolution of the Schrödinger operator and for the representation of the scattering operator S via the *scattering matrix*.

Note that our proof of the asymptotic completeness relies on bound (0.12) with $\beta > 3$ and the spectral condition (19.9), though the results hold under less restrictive conditions, see, e.g., [70].

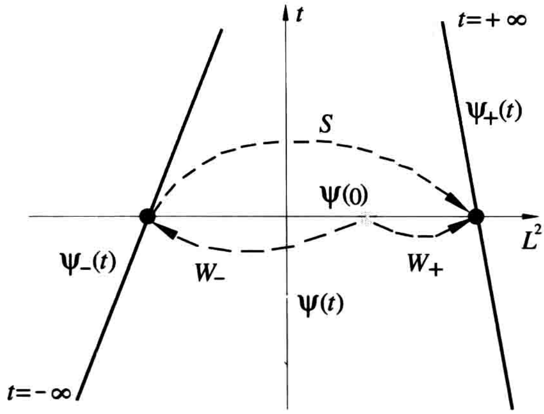

Figure I.1 Scattering and wave operators for $\psi(0) \in X_c$.

In Chapter 9 we apply time decay (0.16) to a dynamical justification of the *quantum differential cross section*. We identify the incident wave with a radiation of a *localized harmonic source* in the Schrödinger equation:

$$i\dot{\psi}(x,t) = H\psi(x,t) - \rho_q(x)\, e^{-iE_k t}\,, \quad H := -\frac{1}{2}\Delta + V(x)\,, \tag{0.21}$$

where $E_k = k^2/2$ for $k \in \mathbb{R}^3$, and $\rho_q(x) := |q|\rho(x-q)$ is the form factor of the source. We assume that the discrete spectrum of H is empty, $\|\langle x\rangle^{\sigma'}\rho(x)\| < \infty$, and $\|\langle x\rangle^{\sigma_0}\psi(x,0)\| < \infty$ with some $\sigma', \sigma_0 > 5/2$. We also assume the Wiener condition

$$\hat{\rho}(|k|\theta) := \int e^{i|k|\theta x}\rho(x)dx \neq 0\,, \qquad \theta \in \mathbb{R}^3\,,\ |\theta| = 1. \tag{0.22}$$

The first step is the proof of the *limiting amplitude principle*, i.e., the long time asymptotics

$$\psi(x,t) \sim B_q(x)e^{-iE_k t}\,, \qquad t \to \infty\,. \tag{0.23}$$

The main result is the convergence of the *spherical limit amplitudes* B_q to the corresponding *plane limit amplitudes* when $|q| \to \infty$. This convergence justifies the (commonly recognized) expression (25.6) for the differential cross section.

In Chapters 10 and 11 we expose our recent results [45, 48] extending the Agmon-Jensen-Kato theory to the Klein-Gordon and the wave equation.

Methods It is well known since Laplace and Heaviside that the long time asymptotics of the solutions to differential equations depend on the smoothness and analyticity of the Fourier-Laplace transform.

The ideas were developed by Vainberg to prove local energy decay (0.6) for general hyperbolic partial differential equations with constant coefficients outside a compact region, and initial functions with compact support [85]–[89]. The Vainberg strategy relies on analytical properties of the resolvent: high energy decay and low energy asymptotics and the limiting absorption principle (a smoothness of the resolvent in the continuous spectrum).

The approach was extended by Jensen, Kato, Murata, and others to prove weighted energy decay (0.7) for the Schrödinger equation with generic potentials of algebraic decay and initial functions from the weighted Sobolev spaces with norms (0.8) (see [1, 35, 36, 62]).

For the Schrödinger equation the Fourier-Laplace transform of the solution is expressed in terms of the resolvent:

$$\tilde{\psi}(\omega) := \int_0^\infty e^{i\omega t}\psi(t)dt = -iR(\omega)\psi(0)\,, \quad \operatorname{Im}\omega > 0\,, \tag{0.24}$$

where the integral converges in $\mathcal{L}^2$ due to the "charge conservation"

$$\|\psi(t)\| = \text{const}\,, \qquad t \in \mathbb{R}\,. \tag{0.25}$$

The resolvent $R(\omega)$ is an analytic operator function. This follows from the Fourier transform in the case $V = 0$ and from the Fredholm Theorem for $V \neq 0$. Spectral representation (0.15) is deduced from the Fourier-Laplace inversion formula

$$\psi(t) = \frac{1}{2\pi i}\int_\Gamma e^{-i\omega t}R(\omega)\psi(0)d\omega\,, \tag{0.26}$$

where Γ is an appropriate contour in the complex plane.

Limiting absorption principle (0.14) in the case $V = 0$ follows by Agmon's bounds [see (0.28) below] and duality arguments. In the case $V \neq 0$ the proof relies on Kato's theorem on the absence of positive embedded eigenvalues and Agmon's theorem on the decay of the eigenfunctions.

Dispersion decay (0.16) is the central point of our lectures. Its proof relies on integral representation (0.15). For $\psi(0) \in X_c$ representation (0.15) becomes

$$\psi(t) = \frac{1}{2\pi i}\int_0^\infty e^{-i\omega t}\Big[R(\omega + i0) - R(\omega - i0)\Big]\psi(0)\, d\omega\,. \tag{0.27}$$

This *oscillatory* integral representation implies time decay (0.16) by the following asymptotics of the resolvent in appropriate operator norms.

A. *High energy decay* of the resolvent and its derivatives:

$$R^{(k)}(\omega) = \mathcal{O}(|\omega|^{-\frac{1+k}{2}}), \quad |\omega| \to \infty;\ \omega \in \mathbb{C} \setminus [0,\infty)\,. \tag{0.28}$$

B. *Low energy asymptotics* at the edge point $\omega = 0$ of the continuous spectrum:

$$R^{(k)}(\omega) = \mathcal{O}(|\omega|^{\frac{1}{2}-k}), \quad \omega \to 0;\ \omega \in \mathbb{C} \setminus [0,\infty)\,. \tag{0.29}$$

The last asymptotics hold under *spectral condition* (19.9) for the Schrödinger operator.

Asymptotics **A** and **B** with $k = 0, 1, 2$ imply dispersion decay (0.16) of the oscillatory integral (0.27) by double partial integration for large ω and by the Jensen-Kato-Zygmund lemma (Lemma 22.5) on "one-and-half partial integration" for small ω.

Asymptotic completeness (0.17), (0.18) for initial functions $\psi(x,0) \in X_c$ with the finite norm (0.8) follows from dispersion decay (0.16) by the classical Cook argument [70]. Namely, the Duhamel representation gives

$$\begin{aligned}
\psi(t) &= U_0(t)\psi(0) - i\int_0^t U_0(t-\tau)V\psi(\tau)d\tau \\
&= U_0(t)\Big[\psi(0) - i\int_0^\infty U_0(-\tau)V\psi(\tau)d\tau\Big] \\
&\quad + i\int_t^\infty U_0(t-\tau)V\psi(\tau)d\tau\,,
\end{aligned} \tag{0.30}$$

where $U_0(t)$ is the dynamical group of the free Schrödinger equation. The integral in the middle line represents $\psi_+(x,t)$ from (0.17). It converges in $\mathcal{L}^2$ since $U_0(-\tau)$ is the unitary operator, while

$$\|V\psi(\tau)\| \le C\|\langle x\rangle^{-\sigma}\psi(x,\tau)\| \le C(\sigma)\langle\tau\rangle^{-3/2}\|\langle x\rangle^{\sigma}\psi(x,0)\| \tag{0.31}$$

by (0.16). The same arguments imply the decay in $\mathcal{L}^2$ of the last integral in (0.30) which represents the remainder $r_+(x,t)$ from (0.17). Hence, (0.17), (0.18) follow for $t \to \infty$.

Limiting amplitude principle (0.23) follows immediately from dispersion decay (0.16) since

$$\psi(x,t) = U(t)\psi_0 + i\int_0^t U(t-s)\rho_q e^{-iE_k s}ds \tag{0.32}$$

by the Duhamel representation. Indeed, the first term on the right hand side converges to zero by (0.16) since the discrete spectrum of H is empty. On the other hand, the second term can be written as

$$ie^{-iE_k t}\int_0^t U(\tau)\rho_q e^{iE_k\tau}ds\,, \tag{0.33}$$

where the integrand decays like $\langle \tau \rangle^{-3/2}$ by (0.16) since $\| \langle x \rangle^{\sigma'} \rho(x) \| < \infty$.

The proof of the convergence of the spherical limit amplitudes B_q to the plane limit amplitudes relies on i) uniform bounds for the spherical amplitudes and the corresponding compactness arguments and ii) the Ikebe uniqueness theorem for the Lippmann-Schwinger equation. We obtain the uniform bounds from the Agmon-Jensen-Kato analytic theory of the resolvent and the long range asymptotics for the Coulombic potentials which are due to Povzner [67], Ikebe [30], and Berezin and Shubin [4].

Let us stress that the limiting amplitude principle (0.23) is a fundamental peculiarity of the hyperbolic partial differential equations (PDEs) which relies on the dispersion decay, i.e., on the energy radiation to infinity.

The extension of the Agmon-Jensen-Kato approach to the Klein-Gordon and wave equations is not straightforward, since the high energy behavior of the corresponding resolvents is quite different from the Schrödinger case (0.28). The difference is related to the distinct nature of wave propagation for relativistic and nonrelativistic equations.

First we prove the long time decay of solutions in weighted energy norms for the corresponding free equations and then extend the decay to the perturbed equations. The proof for the free equations relies on the Strong Huygens Principle in the case of the wave equation and on the corresponding "soft version" of this principle in the case of the Klein-Gordon equation.

The extension to the perturbed equations relies on the Born series and convolution representations.

CONTENTS IN BRIEF