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CLASSICAL THEORY OF ARITHMETIC FUNCTIONS

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Classical Theory of Arithmetic Functions

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Dedicated to the memory of my parents

R. R. Ramakrishna Ayyar

and

T. S. Lakshmy Ammal

Preface

This book is an exposition of certain aspects of the theory of arithmetic functions. The early stimulus to the study of arithmetic functions was provided by the work of R. Vaidyanathaswamy and Eckford Cohen, following the contributions to number theory at the turn of the century by L. E. Dickson, E. T. Bell, G. H. Hardy, and Srinivasa Ramanujan. We present a broad spectrum of results on the classical theory of number-theoretic functions, with emphasis on the role of algebraic and multiplicativity techniques of the theory.

The book presupposes a knowledge of elementary number theory, as in I. Niven and H. S. Zuckerman, *An Introduction to the Theory of Numbers* (John Wiley & Sons). We also assume that the reader has a familiarity with the basic ring-theoretic and group-theoretic concepts and an understanding of the properties of field extensions and finite-dimensional vector spaces.

Divided into three parts, the book is organized as follows: Part 1 includes six chapters which form the core material of the book. Chapter I is on arithmetical convolutions: the Dirichlet convolution, unitary convolution, Cauchy convolution, and Lucas product are discussed, and the algebraic aspects of these binary operations are pointed out. Chapter II analyzes the operation of Dirichlet convolution in greater detail. The proof that the ring of arithmetic functions is a unique factorization domain is patterned after the one due to E. D. Cashwell and C. J. Everett. The distributivity properties given by J. Lambek and Eric-Langford are also included. In Chapter III, the identity for multiplicative functions due to R. Vaidyanathaswamy is established along with its application to quadratic functions. Chapter IV deals with the divisor functions, and the properties of the functions $d_k(r)$ (due to Martin G. Beumer) and $\sigma_k(r)$ are given. Chapter V is an account of the properties of the Euler ϕ-function, and its generalizations and analogs. In Chapter VI the Möbius function is discussed, with particular reference to the genesis of Möbius inversion. Eckford Cohen's notion of direct-factor sets is introduced with a view to giving yet another aspect of the generalization of Möbius inversion.

The four chapters of Part 2 treat arithmetic functions of two variables with special reference to Ramanujan's sum C(n,r), defined by

$$C(n,r) = \sum_{\substack{h(\mathrm{mod}\ r)\\(h,r)=1}} \exp\left(\frac{2\pi i h n}{r}\right).$$

In Chapter VII, we derive the identity for multiplicative functions of two variables due to C. S. Venkataraman and interpret the arithmetical representation of C(n,r). Chapter VIII deals with multiplicative functions connected with a finite abelian group and includes results due to T. Venkatarayudu. It is proved that the number of cyclic subgroups of a group G of order r is equal to the number of divisors of r, if and only if G is cyclic. In this context, we point out the application of Burnside's lemma to a number-theoretic identity due to P. Kesava Menon. Chapter IX is on Ramanujan's sum and its generalizations. A few theorems of Ramanujan are included, and also the reciprocity law for C(n,r) due to Kenneth R. Johnson, and the unitary analog of C(n,r) due to Eckford Cohen. Chapter X is on cyclotomic polynomials. We derive the formula for the discriminant $D(F_n)$ of the nth cyclotomic polynomial $F_n(x)$ using the expressions for the resultant $\rho(F_m,F_n)$ of $F_m(x)$ and $F_n(x)$ due to Tom M. Apostol. It is not generally recognized that the expression for $D(F_n)$ could be given in terms of a determinant (of order $\phi(n)$) in which the (i,j)th entry is $C(i + j - 2, n)$, and so is related to Ramanujan's sum.

Part 3 is a collection of topics distributed over seven chapters. Chapt XI takes another look at multiplicative functions, including the hypomultiplicative functions due to D. B. Lahiri and their relationship to the function $r_s(n)$ denoting the number of representations of n as the sum of s squares. The properties of the semi-multiplicative functions due to David Rearick are also derived. Chapter XII deals with the τ-function of Ramanujan and L. J. Mordell's proof of the multiplicativity of τ(n). Though the theory of Hecke operators superseded Mordell's work, the proof using modular forms of weight 12 is of intrinsic interest. The formula for τ(n) due to John A. Ewell is also included. D. H. Lehmer's result on the primality of τ(n) is proved, showing that τ(n) is composite for $2 \leqslant n \leqslant 63{,}000$ and that $\tau(251^2) = \tau(63001)$ is the first prime value of τ(n). Noting that τ(n) is an example of a specially multiplicative function (a quadratic function), we derive the properties of specially multiplicative functions in Chapter XIII.

Using Aurel Wintner's definition of a limit periodic (B) function, we present the Ramanujan expansions of certain arithmetic functions in Chapter XIV, along with the theorem of H. Delange (1976). In Chapter XV, the notion of a linear associative algebra is introduced, and R. Vaidyanathaswamy's class division of integers (mod r) is discussed. Chapter XVI turns to the work of Eckford Cohen on (r,F)-arithmetic functions, even functions (mod r), and applications to linear congruences. In this connection, Smith's determinant is considered and its generalizations due to Paul J. McCarthy and Charles R. Wall are pointed out. The concluding Chapter XVII highlights the theory of arithmetic functions defined on $GF[p^n,x]$ with reference to the contributions of L. Carlitz, Eckford Cohen, and K. Nageswara Rao.

It is with great pleasure that I acknowledge the help of Professor Paul J. McCarthy, in extensive discussions and suggestions, and revision of the manuscript. In particular, the chapter on Ramunujan expansions of certain arithmetic functions (Chapter XIV) and the chapter on periodic functions mod r (Chapter XVI) are the outcome of detailed consultations with him. The material in Chapters XV and XVII includes ideas received from the late Professor K. Nageswara Rao. On a more personal level, I am grateful to Professor T. P. Srinivasan of the University of Kansas and Professor D. Kannan of the University of Georgia for their great help and encouragement. I owe thanks to Mrs. Sharon Gumm for her superb typing.

I thank the University of Kansas for the opportunity provided to me in 1987-1988 to visit their Department of Mathematics, where this work was completed. The Mathematics Library and the librarian, Mrs. Ruth Fauhl, were of valuable assistance. I have also made use of the library facilities of Tata Institute of Fundamental Research, Bombay, and the Ramanujan Institute for Advanced Study in Mathematics, Madras, and I thank them for their help.

I am grateful to the University Grants Commission, New Delhi, for the financial support under Grant Number 10315, which enabled me to undertake this work. My thanks are due also to the University of Calicut for its support and for the leave of absence granted me to visit the University of Kansas during 1987-1988. Finally, I wish to thank Ms. Maria Allegra of Marcel Dekker, Inc., for her cooperation in the publication of this volume.

R. Sivaramakrishnan

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