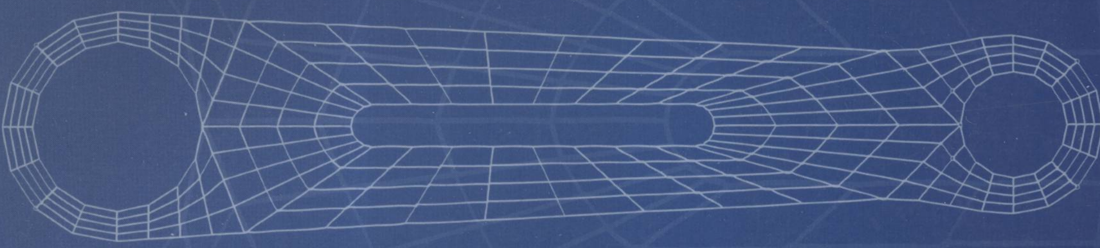
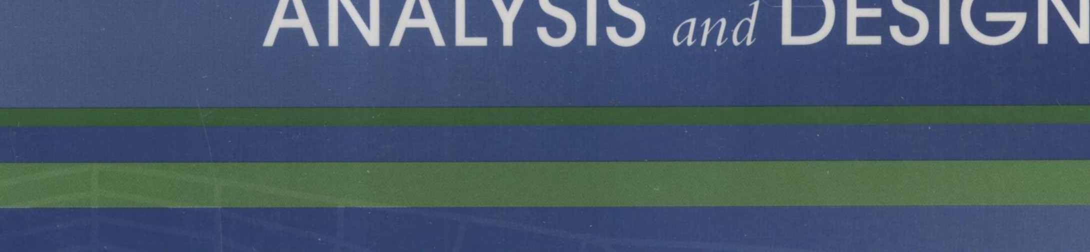


Introduction to FINITE ELEMENT ANALYSIS *and* DESIGN

NAM-HO KIM | BHAVANI V. SANKAR



Introduction to Finite Element Analysis and Design

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Introduction to Finite Element Analysis and Design

To our mothers, Sookyung and Rajeswari

Preface

Finite Element Method (FEM) is a numerical method for solving differential equations that describe many engineering problems. One of the reasons for FEM's popularity is that the method results in computer programs versatile in nature that can solve many practical problems with a small amount of training. Obviously, there is a danger in using computer programs without proper understanding of the theory behind them, and that is one of the reasons to have a thorough understanding of the theory behind FEM.

Many universities teach FEM to students at the junior/senior level. One of the biggest challenges to the instructor is finding a textbook appropriate to the level of students. In the past, FEM was taught only to graduate students who would carry out research in that field. Accordingly, many textbooks focus on theoretical development and numerical implementation of the method. However, the goal of an undergraduate FEM course is to introduce the basic concepts so that the students can use the method efficiently and interpret the results properly. Furthermore, the theoretical aspects of FEM must be presented without too many mathematical niceties. Practical applications through several design projects can help students to understand the method clearly.

This book is suitable for junior/senior level undergraduate students and beginning graduate students in mechanical, civil, aerospace, biomedical and industrial engineering, and engineering mechanics; researchers and design engineers in the above fields.

The textbook is organized into three parts and ten chapters. Part 1 reviews some concepts in mathematics and mechanics of materials that are prerequisite for learning finite element analysis. The objective of Part 1 is to establish a common ground before learning the main topics of FEM. Depending on the prerequisite courses, some portions of Part 1 can be skipped. Chapter 0 summarizes most mathematical preliminaries that will be repeatedly used in the text. The purpose of the chapter is by no means to provide a comprehensive mathematical treatment of the subject. Rather, it provides a common notation and the minimum amount of mathematical knowledge that will be required in future chapters, including matrix algebra and minimization of quadratic functions. In Chapter 1, the concepts of stress and strain are presented followed by constitutive relations and equilibrium equations. We limit our interest to linear, isotropic materials in order to make the concepts simple and clear. However, advanced concepts such as transformation of stress and strain, and the eigen value problem for calculating the principal values, are also included. Since in practice FEM is used mostly for designing a structure or a mechanical system, design criteria are also introduced in Chapter 1. These design criteria will be used in conjunction with FEM to determine whether a structure is safe or not.

Part 2 introduces one-dimensional finite elements, including truss and beam elements. This is the major part of the text that will teach the fundamental aspects of the FEM. We take an approach that provides students with the concepts of FEM incrementally, rather than providing all of them at the same time. Chapter 2 first introduces the direct stiffness method using spring elements. The concepts of nodes, elements, internal forces, equilibrium, assembly, and applying boundary conditions are presented in detail. The spring element is then extended to the uniaxial bar element without introducing interpolation. The concept of local (elemental) and global coordinates and their transformations are introduced via two– and three–dimensional truss elements. Four design projects are provided at the end of the chapter, so that students can apply the method to real life problems. The direct method in Chapter 2 provides a clear physical insight into FEM and is preferred in the beginning stages of learning the principles. However, it is limited in its application in that it can be used to solve one-dimensional problems only. The variational method is akin to the methods of calculus of variations and is a powerful tool for deriving the finite element equations. However, it requires the existence of a functional, minimization which results in the solution of the differential equations. We include a simple 1–D variational formulation in Chapter 3 using boundary value problems. The concept of polynomial approximation and domain discretization is introduced. The formal procedure of finite element analysis is also presented in this chapter. We focus on making a smooth connection between the discrete formulation in Chapter 2 and the continuum formulation in Chapter 3. The 1–D formulation is further extended to beams and plane frames in Chapter 4. At this point, the direct method is not useful because the stiffness method generated from the direct method cannot provide a clear physical interpretation. Accordingly, we use the principle of minimum potential energy to derive the matrix equation at the element level. The 1–D beam element is extended to 2–D frame element by using coordinate transformation. A 2–D bicycle frame design project is included at the end of the chapter. The finite element formulation is extended to the steady–state heat transfer problem in Chapter 5. This chapter is a special application of Chapter 3, including the convection boundary condition as a special case.

Part 3 is focused on two– and three–dimensional finite elements. Unlike 1–D elements, finite element modeling (domain discretization) becomes an important aspect for 2–D and 3–D problems. In Chapter 6, we introduce 2–D isoparametric solid elements. First, we introduce plane-stress and plane-strain approximation of 3–D problems. The governing variational equation is developed using the principle of minimum potential energy. Different types of elements are introduced, including triangular, rectangular, and quadrilateral elements. Numerical performance of each element is discussed through examples. We also emphasize the concept of isoparametric mapping and numerical integration. Three design projects are provided at the end of the chapter, so that students can apply the method to real life problems. In Chapter 7, we discuss traditional finite element analysis procedures, including preliminary analysis, pre-processing, solving matrix equations, and post-processing. Emphasis is on selection of element types, approximating the part geometry, different types of meshing, convergence, and taking advantage of symmetry. A design project involving 2–D analysis is provided at the end of the chapter. Since one of the important goals of FEM is to use the tool for engineering design, Chapter 8 briefly introduces structural design using FEM. The basic concept of design parameterization and the standard design problem formulation are presented. This chapter can be skipped depending on the schedule and content of the course.

Depending on the students' background one can leave out certain chapters. For example, if the course is offered at a junior level, one can leave out Chapters 3 and 8 and include Chapters 0 and 1. On the otherhand, Chapters 0,1 and 8 can be left out and Chapter 3 included at senior/beginning graduate level class.

Usage of commercial FEA programs is summarized in the Appendix. It includes various examples in the text using Pro/Engineer, Nastran, ANSYS, and the MATLAB toolbox developed at the Lund University in Sweden. Depending on availability and experience of the instructor, any program can be used as part of homework assignments and design projects. The textbook website will maintain up-to-date examples with the most recent version of the commercial programs.

Each chapter contains a comprehensive set of homework problems, some of which require commercial FEA programs. A total of nine design projects are provided in the book. We have included access to the NEi Nastran software which can be downloaded from www.wiley.com/college/kim.

We are thankful to the students who took our course and used the course package that had the same material as in this book. We are grateful for their valuable suggestions especially regarding the example and exercise problems.

January, 2008
Nam-Ho Kim and Bhavani V. Sankar

Contents

Chapter 0

Mathematical Preliminaries

Since vector calculus and linear algebra are used extensively in finite element analysis, it is worth reviewing some fundamental concepts and recalling some important results that will be used in this book. A brief summary of concepts and results pertinent to the development of the subject are provided for the convenience of students. For a thorough understanding of the mathematical concepts, readers are advised to refer to any standard textbook, e.g., Kreyszig[1] and Strang.[2]

0.1 VECTORS AND MATRICES

0.1.1 Vector

A *vector* is a collection of scalars and is defined using a bold typeface[3] inside a pair of braces, such as

$$\{\mathbf{a}\} = \begin{Bmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{Bmatrix} \tag{0.1}$$

In Eq. (0.1), $\{\mathbf{a}\}$ is an N-dimensional column vector. When the context is clear, we will remove the braces and simply use the letter "**a**" to denote the vector. The transpose of **a** above will be a row vector and will be denoted by $\mathbf{a}^T$.

$$\{\mathbf{a}\}^T = \{ a_1 \quad a_2 \quad \cdots \quad a_N \} \tag{0.2}$$

By default, in this text all vectors are considered as column vectors unless specified. For simplicity of notation, a geometric vector in two- or three-dimensional space is denoted by a bold typeface without braces:

$$\mathbf{a} = \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix}, \qquad \text{or} \qquad \mathbf{a} = \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} \tag{0.3}$$

where a_1, a_2, and a_3 are components of the vector **a** in the x-, y-, and z-direction, respectively, as shown in Figure (0.1). To save space, the above column vector **a** can be written as $\mathbf{a} = \{a_1, a_2, a_3\}^T$, in which $\{\ \bullet\ \}^T$ denotes the *transpose*. The above three-dimensional geometric vector can also be denoted using a unit vector in each coordinate direction. Let

[1] E. Kreyszig, *Advanced Engineering Mathematics*, 5th ed., John Wiley & Sons, New York, 1983.

[2] G. Strang, *Linear Algebra and its Applications*, 2nd ed., Academic Press, New York, 1980.

[3] In the classroom one can use an underscore (<u>*a*</u>) to denote vectors on the blackboard.

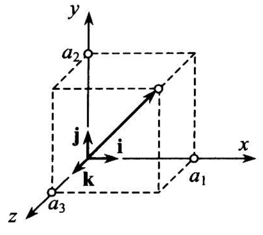

Figure 0.1 Three-dimensional geometric vector

$\mathbf{i} = \{1, 0, 0\}^T$, $\mathbf{j} = \{0, 1, 0\}^T$, and $\mathbf{k} = \{0, 0,1\}^T$ be the unit vectors in the x-, y-, and z-directions, respectively. Then,

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} \tag{0.4}$$

The magnitude of the vector $\mathbf{a}$, $\|\mathbf{a}\|$, is given by

$$\|\mathbf{a}\| = \sqrt{a_1^2 + a_2^2 + a_3^2} \tag{0.5}$$

0.1.2 Matrix

A *matrix* is a collection of vectors and is defined using a bold typeface within square brackets. For example, let the matrix $[\mathbf{M}]$ be a collection of K number of column vectors $\{\mathbf{m}^i\}$, $i = 1, 2, \ldots, K$. Then, the matrix $[\mathbf{M}]$ is denoted by

$$[\mathbf{M}] = [\{\mathbf{m}^1\} \quad \{\mathbf{m}^2\} \quad \cdots \quad \{\mathbf{m}^K\}] \tag{0.6}$$

where

$$\{\mathbf{m}^i\} = \begin{Bmatrix} m_1^i \\ m_2^i \\ \vdots \\ m_N^i \end{Bmatrix}, \qquad i = 1, \cdots, K \tag{0.7}$$

By expanding each component of $\{\mathbf{m}^i\}$, the matrix $[\mathbf{M}]$ can be denoted using the $N \times K$ number of components as

$$[\mathbf{M}] = \begin{bmatrix} M_{11} & M_{12} & \cdots & M_{1K} \\ M_{21} & M_{22} & \cdots & M_{2K} \\ \vdots & \vdots & \ddots & \vdots \\ M_{N1} & M_{N2} & \cdots & M_{NK} \end{bmatrix} \tag{0.8}$$

where $M_{ij} = m_i^j$ is a component of the matrix. The notation for the subscripts M_{ij} is such that the first index denotes the position in the row, while the second index denotes the position in the column. The components that have the same indices are called diagonal components, e.g., M_{11}, M_{22}, etc. In Eq. (0.8), the dimensions of the matrix $[\mathbf{M}]$ are $N \times K$. When $N = K$, the matrix is called a *square* matrix.

A column vector can be considered as a matrix containing only one column. The column vector $\{\mathbf{m}^i\}$ in Eq. (0.7) is an $N \times 1$ matrix.

0.1.3 Transpose of a Matrix

The *transpose* of a matrix can be obtained by switching the rows and columns of the matrix. For example, transpose of the matrix [**M**] in Eq. (0.8) can be written as

$$[\mathbf{M}]^T = \begin{bmatrix} M_{11} & M_{21} & \cdots & M_{N1} \\ M_{12} & M_{22} & \cdots & M_{N2} \\ \vdots & \vdots & \ddots & \vdots \\ M_{1K} & M_{2K} & \cdots & M_{NK} \end{bmatrix} \tag{0.9}$$

which is now a matrix of size $K \times N$.

0.1.4 Symmetric Matrix

A matrix is called *symmetric* when the matrix and its transpose are identical. It is clear from the definition that only a square matrix can be a symmetric matrix. For example, if [**S**] is symmetric, then

$$[\mathbf{S}] = [\mathbf{S}]^T = \begin{bmatrix} S_{11} & S_{12} & \cdots & S_{1N} \\ S_{12} & S_{22} & \cdots & S_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ S_{1N} & S_{2N} & \cdots & S_{NN} \end{bmatrix} \tag{0.10}$$

Matrix [**A**] is called *skew-symmetric* when $[\mathbf{A}]^T = -[\mathbf{A}]$. It is clear that the diagonal components of skew-symmetric matrix are zero. A typical skew-symmetric matrix can be defined as

$$[\mathbf{A}] = \begin{bmatrix} 0 & A_{12} & \cdots & A_{1N} \\ -A_{12} & 0 & \cdots & A_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ -A_{1N} & -A_{2N} & \cdots & 0 \end{bmatrix} \tag{0.11}$$

0.1.5 Diagonal and Identity Matrices

A *diagonal matrix* is a special case of a symmetric matrix in which all off-diagonal components are zero. An identity matrix is a diagonal matrix in which all diagonal components are equal to unity. For example, the (3×3) *identity matrix* is given by

$$[\mathbf{I}_3] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{0.12}$$

0.2 VECTOR-MATRIX CALCULUS

0.2.1 Vector and Matrix Operations

Addition and subtraction of vectors and matrices are possible when their dimensions are the same. Let {**a**} and {**b**} be two N-dimensional vectors. Then, the addition and subtraction of these two vectors are defined as

$$\begin{aligned} \{\mathbf{c}\} &= \{\mathbf{a}\} + \{\mathbf{b}\}, \quad \Rightarrow \quad c_i = a_i + b_i, \quad i = 1, \cdots, N \\ \{\mathbf{d}\} &= \{\mathbf{a}\} - \{\mathbf{b}\}, \quad \Rightarrow \quad d_i = a_i - b_i, \quad i = 1, \cdots, N \end{aligned} \tag{0.13}$$

Note that the dimensions of the resulting vectors $\{\mathbf{c}\}$ and $\{\mathbf{d}\}$ are the same as those of $\{\mathbf{a}\}$ and $\{\mathbf{b}\}$.

A scalar multiple of a vector is obtained by multiplying all of its components by a constant. For example, k times a vector $\{\mathbf{a}\}$ is obtained by multiplying each component of the vector by the constant k:

$$k\{\mathbf{a}\} = \{ka_1 \quad ka_2 \quad \cdots \quad ka_N\}^T \tag{0.14}$$

Similar operations can be defined for matrices. Let $[\mathbf{A}]$ and $[\mathbf{B}]$ be $N \times K$ matrices. Then, the addition and subtraction of these two matrices are defined as

$$\begin{aligned} [\mathbf{C}] &= [\mathbf{A}] + [\mathbf{B}], \quad \Rightarrow \quad C_{ij} = A_{ij} + B_{ij}, \quad i = 1, \cdots, N, \quad j = 1, \cdots, K \\ [\mathbf{D}] &= [\mathbf{A}] - [\mathbf{B}], \quad \Rightarrow \quad D_{ij} = A_{ij} - B_{ij}, \quad i = 1, \cdots, N, \quad j = 1, \cdots, K \end{aligned} \tag{0.15}$$

Note that the dimensions of matrices $[\mathbf{C}]$ and $[\mathbf{D}]$ are the same as those of matrices $[\mathbf{A}]$ and $[\mathbf{B}]$. Similarly, one also can define the scalar multiple of a matrix.

Although the above matrix addition and subtraction are very similar to those of scalars, the multiplication and division of vectors and matrices are quite different from those of scalars.

0.2.2 Scalar Product

Since scalar products between two vectors will frequently appear in this text, it is necessary to clearly understand their definitions and notations used. Let **a** and **b** be two three-dimensional geometric vectors defined by

$$\mathbf{a} = \{a_1 \quad a_2 \quad a_3\}^T, \quad \text{and} \quad \mathbf{b} = \{b_1 \quad b_2 \quad b_3\}^T \tag{0.16}$$

The scalar product of **a** and **b** is defined as

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 \tag{0.17}$$

which is the summation of component-by-component products. Often notations in matrix product can be used such that $\mathbf{a} \cdot \mathbf{b} = \mathbf{a}^T\mathbf{b} = \mathbf{b}^T\mathbf{a}$. If **a** and **b** are two geometric vectors, then the scalar product can be written as

$$\mathbf{a} \cdot \mathbf{b} = \|a\| \|b\| \cos\theta \tag{0.18}$$

where θ is the angle between two vectors. Note that the scalar product of two vectors is a scalar, and hence the name *scalar product.* A scalar product can also be defined as the matrix product of one of the vectors and transpose of the other. In order for the scalar product to exist, the dimensions of the two vectors must be the same.

0.2.3 Norm

The *norm* or the magnitude of a vector [see Eq. (0.5)] can also be defined using the scalar product. For example, the norm of a three-dimensional vector **a** can be defined as

$$\|\mathbf{a}\| = \sqrt{\mathbf{a} \cdot \mathbf{a}} \tag{0.19}$$

Note that the norm is always a non-negative scalar and is the length of the geometric vector. When $\|\mathbf{a}\| = 1$, the vector $\mathbf{a}$ is called the *unit vector*.

0.2.4 Determinant of a Matrix

Determinant is an important concept, and it is useful in solving a linear system of equations. If the determinant of a matrix is zero, then it is not invertible and it is called a singular matrix. The determinant is defined only for square matrices. The formula for calculating the *determinant* of any square matrix can be easily understood by considering a 2×2 or 3×3 matrix. The determinant of a 2×2 matrix is defined as

$$|\mathbf{A}| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21} \tag{0.20}$$

The determinant of a 3×3 matrix is defined as

$$\begin{aligned} |\mathbf{A}| &= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\ &= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ &= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \end{aligned} \tag{0.21}$$

A matrix is called *singular* when its determinant is zero.

0.2.5 Vector Product

Different from the scalar product, the result of the *vector product* is another vector. In the three-dimensional space, the vector product of two vectors $\mathbf{a}$ and $\mathbf{b}$ can be defined by the determinant as

$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \\ &= (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k} \\ &= \begin{Bmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{Bmatrix} \end{aligned} \tag{0.22}$$

In Eq. (0.22), we consider unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ as components of a matrix. As with the scalar product, the vector product can be defined only when the dimensions of two vectors are the same.

In the conventional notation, the vector product of two geometric vectors is defined by

$$\mathbf{a} \times \mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\| \sin\theta \, \mathbf{n} \tag{0.23}$$

where θ is the angle between two vectors and $\mathbf{n}$ is the unit vector that is perpendicular to the plane that contains both vectors $\mathbf{a}$ and $\mathbf{b}$. The right-hand rule is used to determine the positive direction of vector $\mathbf{n}$ as shown in Figure 0.2. It is clear from its definitions in Eqs. (0.22) and (0.23), $\mathbf{a} \times \mathbf{a} = \mathbf{0}$, and $\mathbf{b} \times \mathbf{a} = -\mathbf{a} \times \mathbf{b}$.

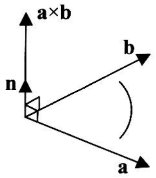

Figure 0.2 Illustration of vector product

0.2.6 Matrix-Vector Multiplication

The matrix-vector multiplication often appears in the finite element analysis. Let [**M**] be a 3×3 matrix defined by

$$[\mathbf{M}] = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix}$$

The multiplication between a matrix [**M**] and a vector **a** is defined by

$$\mathbf{c} = [\mathbf{M}] \cdot \mathbf{a} = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} \cdot \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \begin{Bmatrix} m_{11}a_1 + m_{12}a_2 + m_{13}a_3 \\ m_{21}a_1 + m_{22}a_2 + m_{23}a_3 \\ m_{31}a_1 + m_{32}a_2 + m_{33}a_3 \end{Bmatrix} \tag{0.24}$$

where **c** is a 3×1 column vector. Using a conventional summation notation, Eq. (0.24) can be written as

$$c_i = \sum_{j=1}^{3} m_{ij}a_j, \quad i = 1, 2, 3 \tag{0.25}$$

Since the result of Eq. (0.24) is a vector, it is possible to obtain the scalar product of **c** with a vector **b**, yielding

$$\begin{aligned} \mathbf{b} \cdot [\mathbf{M}] \cdot \mathbf{a} = {} & b_1(m_{11}a_1 + m_{12}a_2 + m_{13}a_3) \\ & + b_2(m_{21}a_1 + m_{22}a_2 + m_{23}a_3) \\ & + b_3(m_{31}a_1 + m_{32}a_2 + m_{33}a_3) \end{aligned} \tag{0.26}$$

which is a scalar.

The above matrix-vector multiplication can be generalized to arbitrary dimensions. For example, let [**M**] be an $N \times K$ matrix and $\{\mathbf{a}\}$ be an $L \times 1$ vector. The multiplication of [**M**] and $\{\mathbf{a}\}$ can be defined if and only if $K = L$. In addition, the result $\{\mathbf{c}\}$ will be a vector of $N \times 1$ dimension.

$$\begin{aligned} \{\mathbf{c}\}_{N\times 1} &= [\mathbf{M}]_{N\times K}\{\mathbf{a}\}_{K\times 1} \\ c_i &= \sum_{j=1}^{K} m_{ij}a_j, \quad i = 1, \ldots N \end{aligned} \tag{0.27}$$

0.2.7 Matrix-Matrix Multiplication

The matrix-matrix multiplication is a more general case of Eq. (0.24). For 3×3 matrices, the matrix-matrix multiplication can be defined as

$$[\mathbf{C}] = [\mathbf{A}][\mathbf{B}] \tag{0.28}$$

where [**C**] is also a 3×3 matrix. Using the component notation, Eq. (0.28) is equivalent to

$$C_{ij} = \sum_{k=1}^{3} A_{ik} B_{kj}, \quad i = 1, 2, 3, \quad j = 1, 2, 3 \tag{0.29}$$

The above matrix-matrix multiplication can be generalized to arbitrary dimensions. For example, let the dimensions of matrices [**A**] and [**B**] be $N \times K$ and $L \times M$, respectively. The multiplication of [**A**] and [**B**] can be defined if and only if $K = L$, i.e., the number of columns in the first matrix must be equal to the number of rows in the second matrix. In addition, the dimension of the resulting matrix [**C**] will be $N \times M$.

$$C_{ij} = \sum_{k=1}^{K} A_{ik} B_{kj}, \quad i = 1, \ldots, N, \quad j = 1, \ldots, M \tag{0.30}$$

EXAMPLE 0.1 ***Determinant***

The reader is encouraged to derive the following results using Eq. (0.20).

$$\begin{vmatrix} a & b \\ 0 & 0 \end{vmatrix} = 0$$

$$\begin{vmatrix} ka & kb \\ c & d \end{vmatrix} = k \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = - \begin{vmatrix} c & d \\ a & b \end{vmatrix} = - \begin{vmatrix} b & a \\ d & c \end{vmatrix}$$

$$\begin{vmatrix} a & b \\ ka & kb \end{vmatrix} = 0, \qquad \begin{vmatrix} a & ka \\ b & kb \end{vmatrix} = 0$$

$$\begin{vmatrix} a+e & b+f \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} e & f \\ c & d \end{vmatrix} = (ad - bc) + (ed - cf)$$

0.2.8 Inverse of a Matrix

If a square matrix [**A**] is invertible, then one can find another square matrix [**B**] such that [**A**][**B**] = [**B**][**A**] = [**I**], and then [**B**] is called the inverse of [**A**] and vice versa. A simple expression can be obtained for the *inverse* of a matrix when the dimension is 2×2, as

$$[\mathbf{A}]^{-1} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1} = \frac{1}{|\mathbf{A}|} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} \tag{0.31}$$

For procedures of inverting a general $N \times N$ matrix, the reader should refer to textbooks such as Kreyszig[4] or Strang.[5] If a matrix is *singular* ($|\mathbf{A}| = 0$), then the inverse does not exist.

[4] E. Kreyszig, *Advanced Engineering Mathematics*, 5th ed., John Wiley & Sons, New York, 1983.

[5] G. Strang, *Linear Algebra and its Applications*, 2nd ed., Academic Press, New York, 1980.