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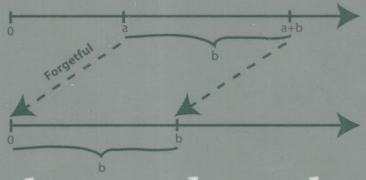

Modeling and Applications to Random Processes

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Probability

To Father and Son

Freddie Ray Miller (1942–1986)
Jacob Anthony Miller

In honor of the man who taught me as a boy and for the boy it is my honor to teach to be a man.

Preface

This text is designed for a one-semester (15–16 week) calculus-based introduction to probability theory and its applications. A unique feature of the book is the synthesis of stochastic models *concurrent* with the development of distribution theory while also introducing the student to basic statistical inference. The text is aimed at advanced undergraduates studying mathematics, statistics, engineering or those students whose discipline is tied heavily to probability and/or random processes. Knowledge of univariate and multivariate calculus is assumed as a prerequisite. Matrix algebra is also necessary for several sections and an undergraduate course in linear algebra is helpful, but not essential for comprehension of the material.

So, why another text on probability? The general theory of probability has been treated by scores of authors and good expositions have been in existence for many years. So, a good book on the subject of probability had better have something unique to offer while still covering the essential theory in a lucid and mathematically sound way. Trying to make probability theory more accessible at the expense of mathematical accuracy is a goal not worth pursuing. Therefore, I strive for mathematical correctness above all. However, students often complain that theory alone is cold and uninviting and that textbook authors tend to echo this by writing in stuffy, pretentious style. While I do not completely agree, this criticism is not without some shred of merit.

What is different here? Two things. First, and foremost, the focus is squarely on probability *modeling*. The power inherent in many probability topics arises through their use as models for random behavior in the physical world. My goal is to continually couch material in the context of creating models for random phenomena. It is my personal belief that the use of probability as a modeling tool hangs like an elusive cloud over many texts and courses in probability. This is

unfortunate. The use of mathematical, probabilistic and statistical models should be something that instructors, departments and programs in the mathematical sciences should triumph and accent. The model building process and the use of mathematical structures as models are still underrepresented in our undergraduate curriculum which trains mathematicians, statisticians and engineers. This should change.

Second, opportunities have been missed to introduce students to the beauty and applications of stochastic processes while they are learning introductory probability theory. It is common to find a chapter dealing with stochastic processes stuck and the end of a probability text. While the coverage may be appropriate and accessible, such positioning reduces stochastic processes to an after thought. It suggests that the link of probability to stochastic models is optional and extra. This is also unfortunate. My goal is to synthesize the major stochastic processes with the obligatory coverage of probability laws, random variables and distribution theory. This serves two purposes. On one hand, it provides a student with an immediate application of the fundamental theory. On the other, the student acquires knowledge of the major stochastic processes. Thus, the student is better equipped and empowered to be a modeler, a true problem solver and a critical thinker.

Elementary statistical inference is introduced throughout the text as well. Many students leave a course in probability theory without a proper understanding that parameters of probability models are often unknown in practice. Statistical science cannot exist without probability theory and hence, basic estimation techniques as well as the philosophy of hypothesis testing are scattered continuously throughout the latter chapters of the book. Indeed, mastery of the material in this text puts the student in a perfect position for a follow-up course on the theory and applications of statistical inference (or a course typically called 'mathematical statistics').

A conversational writing style is used frequently in the chapters that follow. Unfortunately, mathematicians and statisticians are occasionally labeled as inaccessible, lofty types who cannot bear to speak the language of those they hope to influence with their knowledge. It is especially common to label academicians in this way. Do we deserve such criticism? Partly so, I believe. Once those trained in the mathematical sciences interact and develop working relationships with practitioners in other disciplines, the stereotypes of our profession are then often thwarted. Nonetheless, those who profess in the mathematical sciences need to be better communicators of the material from our discipline. This can start with the way we write. This is scary and hard. We are uncomfortable taking a conversational tone in writing because we, in part, feel as though we are "selling-out" and are risking becoming mathematically imprecise. Why? Conversational, warm inviting tones in writing can surely co-exist with mathematical precision. This is my aim.

To the Student

The mastery of the material in a probability course depends predominantly upon your dedication to study and the quality of delivery provided by your instructor. Make no mistake about it, the primary set of learning responsibilities always rests with the student as opposed to a book or instructor. A textbook should be a secondary, yet vital, recipient of a student's attention. Your primary attention should be directed to your class and the material discussed while you are face-to-face with your instructor. Your instructor has life, has inflection, uses body language, and can talk back to you when questioned. This text possesses no such qualities. Use this book as a vehicle to accent and generate discussion between you, your instructor and your classmates.

Despite your instructor being the primary source of material, this text was written to be read thoroughly. As stated in the preface, I purposely use a conversational writing style in many places. It is my belief that textbooks *can* be written in a warm, inviting tone while simultaneously being mathematically correct. Examples and proofs are presented with a great deal of supporting English structure—even in the midst of topics which are heavy on notation. This is done purposefully with the hope that it will encourage you to develop your mathematical communication skills.

Don't fall victim to the dreaded "zebra effect" while studying from this book. That is, don't make the only pages soiled on the sides with oil from your hands those that contain the exercises. Pick up a textbook you've used in another class and look at it closely from the side and you'll see what I mean. Are the sides of the pages all white? Then it wasn't read enough. Are there white stripes followed by gray/black ones (zebra effect)? Then the pages which contain exercises were

those pages most examined and you should have spent more time actually reading the body of the text. Your goal is to make the sides of all the pages turn gray.

As a student in a discipline that makes heavy use of mathematics, it is not enough for you to just *know* a fair amount of material. The mathematics you learn only empowers you if you can *communicate* what you have learned. Effective communication of mathematics requires good style. You may have a really good mathematical idea, but when it is presented in a sloppy, disjointed manner you have essentially rendered your good idea meaningless. So, I encourage you: Take the opportunity to learn how to write, speak and read mathematics better than what you do now. You may already be a good expositor of your mathematical ideas. There is *always* room for improvement—and that goes for all of us!

This book is focused on using probability to model the random behavior that exists around you in the natural world. Probability is the branch of mathematics which quantifies the level of uncertainty that we encounter in every day decision making. Becoming a better decision maker is alone a sufficient reason for studying probability. The great mathematician René Descartes said, "When it is not in our power to determine what is true, we ought to act accordingly to what is most probable...". The bottom line is that very, very little in our lives can be predicted with certainty. Uncertainty is the reason for a great deal of fear, emotion, worry, and nervousness. Yet, the uncertainty we face gives us reason for hope, cultivates a sense of eagerness and leads to an anticipation of "what might be".

Knowledge of basic probability really can add to the quality of your life. We are in control of so little. If we possess some knowledge of probability, we will be able to understand more about the physical world and the randomness that we see around us. More than that, we may be able to alter in a positive manner some of life's circumstances. After all, becoming a better decision maker is bound to add to the quality of our life and create more beneficial opportunities for us as we get older.

As you move through the material in this book, answer any questions posed to you in the body of the text. If I write that I have "left it to the reader to show"—then that means you! Read actively and slowly. Review the examples often and work as many exercises as you can. At the end of each chapter you will find a set of terms to review before reading the next chapter. Also, those terms which will reappear later in the book are listed in a section titled "foreshadowing". You should review the conceptual questions that close out each chapter. These questions give you a chance to develop your written and oral communication skills.

Finally, you may not think so, but this text has been written with you in mind. I don't know you, but conceptually I have thought about you. I have written exercises, examples and paragraphs while trying to see through your eyes and reason from your point of view. At least in an abstraction, I have pondered your reactions and concerns as you encounter this material for the first time. You

should feel free to send me an e-mail message at miller@math.sfasu.edu. If you have a kind word to share, I'd love to hear it. If you have a constructive criticism, then pass it along. Authors must absolutely be open to suggestions. I welcome them—all in the spirit of helping you to better comprehend this wonderful and powerful subject.

To the Instructor

I trust you. It is the right thing to do and it is what I want authors to do to me when I use their books. What am I talking about? Well, I think at times authors attempt to be everything to everybody. Occasionally, textbooks include a plethora of chapters about some really great stuff, but are so packed with material that an instructor couldn't possibly address the topics in a time frame imposed by a college semester or term. Have you ever taught from a text in which the last third of the book doesn't see the light of day? I have no such desire to include so many chapters as to try and be everything to everyone by writing this text.

This text is a one-semester (15–16 week) introduction to calculus based probability and its applications. There are no extra chapters at the end of the book that I hope various instructors will choose from at random with which to finish their course. If you have a favorite topic that I don't cover—include it. I don't see any point to writing 500 additional pages to tack on to the end of the text which attempt cover all the pet topics that instructors which use my text might want to deliver to students. You don't need me to write 17 chapters and 1000 pages in order to put you in a position to deliver a good probability course. That's what I mean when I say I trust you.

A few pages back I have told the students that the primary source of material for a good course in probability comes from you, not from me. I mean that wholeheartedly. That's not slacking my responsibility as an author. That's just stating the obvious. You are more important than me when it comes to your course. You have the greatest opportunity to impact their lives since you communicate directly with them. College students remember great instructors far more often than they remember great probability textbooks. Good texts are essential for proper learning, but it is *people* that best transmit knowledge. People such as

yourself are the entities with the greatest chance to bring the material alive and make an experience meaningful for a student.

Let's be honest. It's rare that any instructor views a textbook as flawless in its delivery, style, coverage and order. But, there is one truth here. A good instructor can always circumvent the slight shortcomings that they see in a text assigned to their students to read. However, it is much harder for a text to cover for any flaws in your delivery. You've got the opportunity to make your classroom vibrant and a laboratory of true investigation, comprehension and a place where mathematical maturity is obtained. You've also got the chance at offering your students much less. That's why I trust you—and why I've worked hard to write a book that is readable, informative, mathematically precise and complementary of your delivery. It is my desire to partner with you, not to *be you.*

The major results of introductory probability theory are presented with plentiful examples and exercises. A lucid exposition is mixed with a warm tone. The basics of stochastic processes are uniquely woven amidst the distribution theory which forms the heart of the book. The coverage of the chapters appears in summary form a few pages over.

I have included 395 exercises comprised of 916 separate parts which ask a total of over 1000 different questions across eight chapters. Ninety additional conceptual questions finish out the chapters. I encourage you strongly to be liberal in your assignment of these problems. The exercises are almost universally written in the context of a physical setting. There are very few problems written in the vein "If X is a normal random variable with mean 20 and variance 5, find the probability X exceeds 32." The idea is that the student needs to read information describing a physical setting and then choose a *model* en route to making a computation. Following the exercises, I have included sections which provide terminology for your students to review, terms which foreshadow future topics and a list of conceptual questions. The conceptual questions may be starting places for classroom or office discussions or may be representative of free-response style examination questions.

Finally, I welcome your critiques and suggestions. Please feel free to communicate with me by e-mail at miller@math.sfasu.edu. Any error of any kind is the sole responsibility of the author. The fact that you are reading this implies that you and I share the common desire to teach probability to our students—and to do it well. I welcome any comment that will help me in that quest.

Coverage

The material in this book has been used as the basis of an undergraduate probability course at Stephen F. Austin State University (SFA) for several years. As a rough timeline for coverage, it is suggested that the first four chapters be covered within the first five or six weeks of a typical 15–16 week semester. Section 4.4 can be omitted without loss of continuity or assigned to students as projects and/or extra reading. I believe that a course taught from this text should revolve around Chapters 5, 6 and 7 and that this material comprise one-half to two-thirds of a semester. In our course at SFA, we typically allot nine to ten weeks for these three chapters. The final chapter can be covered in the remaining one or two weeks of the course.

The first chapter sets the tone for the text by introducing the modeling process. Various branches within the mathematical sciences where modeling is prevalent are showcased. The chapter ends with applications to a stochastic process: the random walk. The second chapter reviews basic set operations and the notion of function. The axioms of probability are introduced followed by a section on the combinatorics necessary in equally likely sample spaces.

The major laws and rules associated with calculating probabilities comprise Chapters 3 and 4. The complement rule and addition rule are developed in the very short third chapter and then conditional probability is thoroughly motivated in the first section of Chapter 4. Ample time is given to the theorem of total probabilities due to its importance in applications from stochastic processes. The fourth chapter ends with some challenging problems which can be used in classroom discussions or for projects. At SFA, I frequently assign these problems to groups of students and require them to give an oral presentation to classmates.

Chapter 5 provides the language of random variables and stochastic processes. This chapter contains the basic terminology that one must master to comprehend

distribution theory. Discrete random variables are the subject of Chapter 6. The unique identity of the text comes to fruition at this point. Each probability distribution is covered as a model of random behavior. Statistical estimation of parameters is discussed along with a hook to a major stochastic process. The random walk and binomial distribution are paired. Naturally, the Poisson process follows the introduction of the Poisson distribution. Topics from queueing and birth processes which are linked to the geometric and negative binomial distributions round out the chapter.

The synthesis of distribution theory and stochastic models introduced in Chapter 6 appears again in Chapter 7 in the context of continuous random variables. The relationship between the Poisson process and the continuous uniform and exponential random variables provides a set of applications in the first two sections. More queueing examples are linked to the gamma random variable and the natural connection between Brownian motion and the normal distribution is explored as well. As with the models in Chapter 6, estimation of parameters is discussed. A specific section on the popular chi-squared goodness-of-fit test is included as a way of introducing the philosophy of hypothesis testing. Limit theorems, an important subject when studying probability, are also treated in this chapter.

Chapter 8 completes the text's coverage by examining multivariate random variables with a focus on the concepts of covariance and correlation. Even in this final chapter, applications to stochastic processes are interwoven with Brownian motion reappearing and stationary processes introduced.

The exercises contain additional material that expand the list of topics to which students can be introduced. Instructors should assign the appropriate exercises if they have specific desire to expose their students to these ideas. In addition to the material in the eight chapters, the following probabilistic ideas are presented in the exercises: conditional independence, Simpson's paradox, acceptance sampling, geometric probability, simulation, exponential families of distributions, as well as many non-standard probability distributions. Jensen's inequality is contained in the exercises as well. Chapter 7 contains an entire suite of exercises that introduce the student to the problem of obtaining the distribution of a function of a random variable. See Exercises 7.68–7.79.

The following additional concepts from stochastic processes are touched on in the exercises: first passage times, arcsine and hitting time theorems for random walks and Brownian motion, Little's laws from queueing, Markovian networks, alternating renewal processes, the Brownian bridge and the concept of differencing a time series. Finally, to increase exposure to statistical topics, I have included the sign test, Fisher's exact test, the concept of mean squared error and the creation of confidence intervals using t-statistics amidst the exercises. Most of the exercises referenced in this and the previous paragraph are titled so they can be easily identified.

Acknowledgments

What fun this was! In all honesty, this book was a joy to write. My positive memories are due to many sources: family, colleagues, students and hobbies.

This project could not have been completed without the support of my mathematics and statistics colleagues at SFA. My department is one which cultivates creativity in an environment of congeniality. For that, I am truly blessed and thankful. Jasper Adams, my chair and mentor, provided encouragement and support that are greatly appreciated. I would not have professionally developed to the meager levels that I have without Jasper's help. A special thanks goes to Joe McWilliams and James Stamey, for their many conversations about the nuances of writing. They both read excerpts from the text and provided valuable feedback. Joe's experience as an author gave us a common ground on which to discuss successes, failures, trials and tribulations.

Thanks to Sarah Stovall, for her help with Adobe Illustrator®, and to Bob Feistel, for keeping me equipped with state-of-the-art computers, software and support that helped make the writing process much easier. Some former students also deserve special mention. Thanks to Sarah Horton, for her help in researching articles on Markov chains. Tracey White and Marcus Arreguin read the text with sharp eyes and made many comments which enhanced the final presentation. My MTH 419 class at SFA during the Spring semester of 2005 was the first group to use a pre-print copy of the full textbook. Their comments and collective spirit affected me greatly. Each student from that class deserves credit for their role in the final product and so their names appear throughout the book. Thanks for the autographed baseball, guys! Classes that came before them used notes that eventually blossomed into early drafts of this text. The students from those classes deserve thanks as well.

Most of us have been taught by people who have had a lasting influence on our lives. For me, these people include, but are not limited to: Narayan Bhat, Bill Schucany, Wayne Woodward, Richard Gunst, Pat Odell, Ray Cannon, and Danny Turner. Each of these folks receives my sincere thanks along with the scores of other colleagues, teachers and mentors that have played a role in my academic life.

Most authors are collectors of books themselves. I am no exception. The influence of other books and of other authors runs deep. The works by Feller (1957, 1966), Karlin and Taylor (1975), and Ross (1996, 2000) have had a profound impact on my professional life—and certainly the lives of scores of other probabilists as well.

The team at Wiley has been helpful, responsive and supportive. I knew they would be since we have been through this together before. Specifically, Susanne Steitz and Steve Quigley deserve a round of applause. I appreciate their comments and patience.

Several anonymous reviewers contributed to the quality of the final product delivered here. I'd like to thank each of them for their professionalism and suggestions.

Various versions of Mathematica®, Maple®, Adobe Illustrator®, R statistical software®, JMP®, ITSM® and MathType® were used to create the figures, illustrations, tables and equations in the original drafts of this manuscript.

Like another author of probability books (Rick Durrett), music provided the background for many, many hours of writing and editing for me. Midway through the two-and-a-half years it took to complete this project, a new subwoofer was ordered for the sound system attached to my office computer. Also, my brother purchased me a CD rack large enough to hold my collection. My inspiration level climbed along with the volume at that point. My family, colleagues and students know that I am a connoisseur of progressive hard rock and metal music. So, I should thank the bands Dream Theater, Enchant, Shadow Gallery, Magnitude 9, and others for their 'rock-solid' musical inspiration. I'd be remiss if I didn't mention the comforting rhythms of Rush, the greatest progressive band of all-time, whose permanent waves of sound could almost move pictures on my office walls at times. The loyal fan will get the twist of phrase.

Special thanks without any attempt at explanation goes to Baseball Tonight, Beckett.com, Yahoo! fantasy baseball, the San Diego Padres, eBay®, Dr. Pepper®, the 2004 Baylor Bear football team which beat Texas A&M, attendees at Friday Forum, The Man Club, Smokehouse, and Mr. Will's Restaurant.

Finally, I have saved the best for last. My parents and family have taught me the importance of faith in Christ and what it means to achieve and maintain a real zest and joy for life. My inner passion is driven by their fuel. Christine, you are the most patient person I know. The effect of your being a wonderful wife and mother is something I know I can't accurately estimate. Instead, I'll simply say thanks—and that I don't deserve, but am incredibly thankful for your presence in my life.

Jacob, you may never know how all the whiffle ball games, snow cones, and nights sitting on the couch rooting for the Padres have helped me. You made me forget all the petty little inconveniences and struggles of writing. Reading books to you each night helped me see the relative unimportance of the hundreds of pages that follow in comparison with the hundreds of thousands of memories you have already created for me in your first seven years of being my son. Jacob often claims he will open a restaurant when older—his uniquely named future business venture is motivation for Exercise 6.1.

My mother, Nancy, quite the expositor of the written word herself, has been a constant support. Not for just the two-and-a-half years of writing this book, but for all of the thirty-seven I've been alive. You don't get to choose your parents, but I sure am glad God gave me you and Dad. Additionally, the first chapter of the book reads better due to Mom's suggestions. My brother Doug will relate to some of the acknowledgments in the earlier paragraph slighting explanations. As always, time spent with you is time spent happy. I am proud of you for who you are and how you live. You inspire me and I am thankful for what you add to my life.

This book is written to champion and promote the trait of perseverance. Hebrews 12: 1–2.

Nacogdoches, Texas