ISNM
Vol. 128

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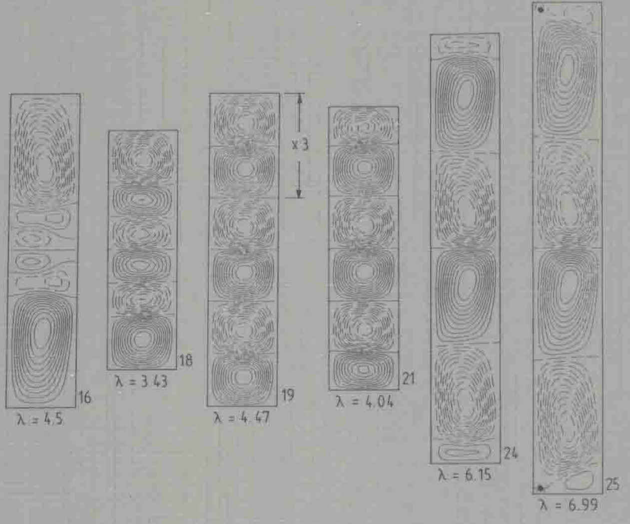

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Birkhäuser Verlag
Basel · Boston · Berlin

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1991 Mathematics Subject Classification 35Qxx, 65Nxx, 76Exx

A CIP catalogue record for this book is available from the Library of Congress, Washington D.C., USA

Deutsche Bibliothek Cataloging-in-Publication Data

Meyer-Spasche, Rita:
Pattern formation in viscous flows : the Taylor-Couette problem and
Rayleigh-Bénard convection / Rita Meyer-Spasche. – Basel ; Boston ;
Berlin : Birkhäuser, 1999
(International series of numerical mathematics ; Vol. 128)
ISBN 3-7643-6047-X (Basel ...)
ISBN 0-8176-6047-X (Boston)

Printed on acid-free paper produced of chlorine-free pulp. TCF ∞
Cover design: Heinz Hiltbrunner, Basel
Printed in Germany
ISBN 3-7643-6047-X
ISBN 0-8176-6047-X

9 8 7 6 5 4 3 2 1

ISNM
International Series of Numerical Mathematics
Vol. 128

Preface

> It seems doubtful whether we can expect to understand fully the instability of fluid flow without obtaining a mathematical representation of the motion of a fluid in some particular case in which instability can actually be observed, so that a detailed comparison can be made between the results of analysis and those of experiment.
>
> – G.I. Taylor (1923)

Though the equations of fluid dynamics are quite complicated, there are configurations which allow simple flow patterns as stationary solutions (e.g. flows between parallel plates or between rotating cylinders). These *flow patterns* can be obtained *only in certain parameter regimes.* For parameter values not in these regimes they cannot be obtained, mainly for two different reasons:

- The mathematical *existence* of the solutions is parameter dependent; or
- the solutions exist mathematically, but they are not *stable.*

For finding stable steady states, two steps are required: the steady states have to be found and their stability has to be determined.

Only a few laminar flows correspond to explicitly known solutions of the equations of motion. In the case of the Taylor problem, for instance, only Couette flow (1.12) is explicitly known. Even fewer flows than explicitly known are simple enough to allow the detailed analysis of their stability with the methods of mathematical physics. The recent development of large computers and of numerical methods makes it possible to solve the full nonlinear equations for many parameter values and to investigate the stability properties of the computed steady states. Scientific computing created completely new possibilities. Nevertheless, only all available methods, i.e.

- experiments with fluids;
- explicit solution of the equations, when possible, and perturbation analysis in a neighborhood of such solutions;
- numerical solution of the full nonlinear equations and numerical stability analysis

together give a more or less complete picture. In the following text we will see again and again how results on the Taylor problem obtained with different methods complement and supplement each other.

It is one of the merits of G.I. Taylor that he saw which aspects of flows between rotating cylinders can or must be neglected to obtain a simplified model which is accessible to mathematical analysis. In his work of 1923 he obtained calculated as well as measured results – impressive for their vastness, completeness and accuracy. Even more impressive are the comparison and good agreement (less than 5% deviation of the calculated from the measured values, with about 2% errors in the measurements).

Since then, the '*Taylor problem*' is a popular research subject, because it is so simple and at the same time so complex. The investigations discussed in the following were performed by mathematicians, physicists and engineers. Contact and exchange between disciplines is typical for work on the Taylor problem: from the beginning, results obtained by theoretical investigations and by experiments were compared to each other, and this comparison was always considered important [Tay, AH92]. Often the Taylor problem was a test case for newly developed methods, and it initiated and stimulated the development of new methods.

For *mathematicians* it is one of the popular examples, especially:

- for studying bifurcations (pitchfork, Hopf and homoclinic), hysteresis and catastrophes; and
- for the development and testing of new methods (analytical and numerical).

Because of the successful cooperation with other disciplines it is a good example of *Applied* Mathematics.

Physicists use it especially for studying:

- the laminar – turbulent transition and
- the occurrence of instabilities owing to the dynamical effects of rotation or (more basically) the occurrence of instabilities of fluid flow.

Engineers are especially interested because of *practical applications*:

- In bearings, an axle or pin rotates in a liquid. This liquid is confined or flows in axial direction. Because of the rotating axle it is practically impossible to keep such a configuration perfectly closed to vacuum. This is why the bearings of satellites, of space shuttle etc. do not simply contain oil, but a 'magnetic suspension': a mixture of plastic-coated iron particles and oil. A magnetic field then confines the iron particles, and surface effects force the oil to stay with the particles [Stie].

- In turbines there is a roughly similar situation, but a blade turns and the flow through the configuration is important.
- For separation of bio fluids, e.g. for separation of liquid and corpuscles of blood, stationary Taylor vortices with radial through flow have been used: the liquid passes through the porous outer cylinder, the corpuscles stay in the 'sieve'. The vortex movement cleans the outer cylinder and keeps it from filling up. In this process the blood corpuscles do not get damaged as much as in the older centrifugal process.
- Also, stationary Taylor vortices with radial through flow and a porous outer cylinder have been used for washing wool.
- During chemical reaction processes it is important that the flows passing each other have well-defined speeds, in order that the reacting fluids mix in well-defined mass ratios. The control of speed via control of Reynolds number is relatively simple. In the Taylor apparatus it is achieved by control of the angular speeds of the cylinders. The mixing of the reacting fluids happens at the boundaries of the wavy vortices: the local velocity changes considerably at these interfaces and this is favorable for mixing. In the interior of the vortices there is not much mixing – the diffusion into the interior of the vortices is too slow [SKC].

In the following we will consider only the simplest basic configuration of the Taylor problem: a Newtonian fluid with constant density, viscosity and temperature between concentric cylinders with periodic boundary conditions in the axial direction. There is a considerable amount of research on variations of this basic configuration [AH92, Roe85, Hol83]:

- short cylinders with rigid lids on top and bottom (boundary effects are not neglected): the Benjamin problem;
- variation of the geometric configuration: concentric spheres or ellipses, non-concentric cylinders, variation of gap width with length, etc.;
- additional through flow in the radial or axial direction;
- additional temperature gradient in the radial direction;
- variation of the fluid: non-Newtonian fluids like magnetic suspensions, polymer solutions, liquid crystals, etc.

The earth and also other planets rotate. Vortex rings are sometimes visible in their atmosphere (rising air over the tropic rain forests, sinking air over the desert belt of the earth). In this case there is an additional temperature gradient which must not be neglected. For studying meteorological questions another model problem is thus relevant, the *Bénard problem*: a fluid in a gravitational field between parallel plates or spheres of different temperatures. According to more recent insights, the compressibility of the atmosphere is more important than its viscosity [Lo93a]. This reduced the importance of the Bénard problem for meteorology. But it is still important for the investigation of flows in other geophysical fluids (in the oceans, in the liquid core of the earth etc.) and also in astrophysics. Besides this, it is a very important model problem for the same basic questions for which the Taylor problem is studied.

There is an abundant amount of literature on both problems. Here only books and review articles are listed:
Taylor problem: [Cha, DR, DPS, Stu, Wim, BC86, AH92, Do92, CI94];
Bénard problem: [Cha, DR, Bus85, Bus89, Strau].

The reviews on the Taylor problem mostly concern experiments and results obtained with perturbation methods and with methods of mathematical physics. The rich literature on numerical investigations consists of articles in journals and in proceedings volumes. They are not adequately mentioned in general reviews. To my knowledge there is no general review that adequately integrates those results which were obtained by numerical methods.

In the following text, we will discuss the Taylor problem in great detail. When discussing the Rayleigh-Bénard problem we will concentrate on those aspects which are of importance for understanding the Taylor problem.

I am very grateful to all who gave me the opportunity for joint work and/or intensive discussion on the topics of this book. Special thanks are due to *Herb Keller* and *Philip Saffman*, who introduced me to the Taylor problem, to *Fritz Busse*, who taught me that I should look at the Rayleigh-Bénard problem to understand the Taylor problem better, to *Dietrich Lortz*, to *Frank Pohl*, who did the programming for the (until now unpublished) investigations of the 9×9 model in Chapter 4, to *Eva Sombach*, who made many figures look better, and to *John Bolstad*, who critically commented on the whole manuscript.

Contents

Chapter 1

The Taylor Experiment

In this chapter we discuss the Taylor experiment and mathematical modeling of it. We review experimental results and investigations of the stability of Couette flow. Numerical modeling of Taylor vortex flows will be discussed in the next chapter.

1.1 Modeling of the experiment

1.1.1 Introduction

In homogeneous fluids instability often is caused by dynamic effects of rotation. An important example for this type of centrifugal instability is flow between rotating cylinders. The instability of rotating fluid was first investigated by **Lord Rayleigh** (1880), in the case of **inviscid fluids** ($\nu = 0$).

Though the assumption $\nu = 0$ seemed to be a straightforward simplification in physics, it has strong consequences for the mathematical nature of the describing equations: they are then the (historically older) Euler fluid equations instead of the Navier-Stokes equations. For $\nu = 0$ the second derivatives in (1.3)–(1.6) are not present, and thus part of the boundary conditions for the full system cannot be satisfied. If the problem is further simplified by considering only circumferential flows $(0, v(r), 0, p(r))$, r being the distance from the axis of rotation, flows with arbitrary circumferential velocities $v(r) = r\Omega(r)$ are possible.

Heuristic physical reasoning led Rayleigh to derive his famous stability criterion, see for instance [DR, p. 71ff]. He also noticed an analogy to gravitational instability of stratified fluids at rest (⇒ Rayleigh-Bénard convection, see below).

Rayleigh's Circulation Criterion: *Necessary and sufficient for stability of an axisymmetric flow of angular velocity $\Omega(r)$ with respect to axisymmetric disturbances is:*

$$\Phi(r) := \frac{1}{r^3}\frac{d}{dr}\Big(r^2\Omega(r)\Big)^2 \geq 0. \tag{1.1}$$

If a flow is called stable if the kinetic energy of axisymmetric, z-periodic perturbations is bounded for all admissible initial conditions, this criterion can be proved mathematically [Cha, §67], [Lo93b].

The stability of viscous ($\nu \neq 0$) rotating fluids was first investigated by chance. In 1881, **M. Margules** [Marg] in Vienna suggested to measure the viscosity ν of a fluid by putting it into the gap between two concentric cylinders, rotating one of them, and measuring the torque exerted onto the other cylinder. **Couette** (1890) measured the viscosity of water of different temperatures by this technique and found values close to the values known today. For small angular velocities he observed the expected linear relation. For larger angular velocities he observed a different inclination. This was interpreted as onset of turbulence (see Figs. 1.2 and 1.3 below and eqs. (1.14) and (2.86)).

Mallock (1888, 1896) independently performed similar experiments. Though he also focused on measuring viscosity, **Lord Kelvin** used his experiment to get insights on the stability of viscous flows between rotating cylinders (letter to Rayleigh [Do92]). Later on **G.I. Taylor** looked at Mallock's experiments under the viewpoint of stability [Tay]. He found that Mallock's results partially contradicted Rayleigh's criterion (1.1) for inviscid flows and that it seemed practically impossible to deduce any rules from these experiments. Taylor analyzed and criticized Mallock's experiments very carefully and pointed out the following possible sources of error:

a) length and diameter of the outer cylinder were nearly of the same size (ca 20 cm);
b) Mallock used the full lengths of the cylinders for his measurements;
c) one of the two cylinders was probably not held rigidly enough and could perform small lateral movements.

Mallock attempted to construct his three apparati such that 'the water' in the gap 'is very nearly in the same condition it would be if' the two cylinders 'were infinitely long' [Do92]. He substituted the original bottom by one of liquid mercury in the course of his work. This suggests that he himself noticed that influences of the bottom were a problem. Taylor guessed that a) and b) led to the result that Mallock's experiment featured instability by 3-dimensional disturbances favored by the bottom and was thus dominated by end effects [Tay]. He found new approaches such that these problems were carefully avoided and such that his experiment could actually be modeled mathematically with periodic boundary conditions in infinitely long cylinders. Also, he constructed his apparatus such that both cylinders could rotate independently. Thus a wider parameter regime became accessible.

In dimensional variables, the problem has the following quantities to be chosen by the experimenter:

R_1 radius of the inner cylinder;
R_2 radius of the outer cylinder;
H total length of the cylinders;
L_o length of that portion of the cylinders where the measurements are made;
Ω_1 angular velocity of the inner cylinder;
Ω_2 angular velocity of the outer cylinder;
ν kinematic viscosity of the fluid.

Experimenters usually give the **aspect ratio**

$$\Gamma := H/D, \quad D := R_2 - R_1 \tag{1.2}$$

of their experiment.

Following Taylor, the Taylor problem in the strong sense has theoretically **infinitely long cylinders**, and the experimenters using Taylor's model usually made sure that the published results do not depend on the aspect ratio. To achieve negligibly small end effects,

- the aspect ratio Γ is made large (Taylor used $\Gamma \geq 90$);
- measurements are performed only in the middle portion of the cylinders, far away from top and bottom (Taylor used 20 cm out of 90 cm);
- the lids at top and bottom are designed in a sophisticated way.

Today, the most advanced way to mimic infinitely long cylinders is using **ramps** at the end portions of the cylinders: the radius of the inner cylinder may be increased or the radius of the outer cylinder may be decreased towards the ends, with optimized length and inclination angle [AC, RP87].

Also, the influence of other effects which are not included in the mathematical model but which could be found in a real experiment must be kept small enough to be negligible. They are of two types: simplifying idealizations and unavoidable imperfections. Idealizations typically are eliminated when science gets more advanced and they turn out to be too unrealistic; imperfections get reduced with the advance of technology. Depending on the questions to be investigated, it is not always clear if a deviation mathematical model – real experiment has to be classified as idealization or imperfection. We give a few examples for both, in the framework of the questions to be investigated in the following text.

Idealizations are for instance:

the fluid is inviscid (eliminated);

the cylinders are infinitely long, i.e. end effects can be neglected;

the fluid is homogeneous, i.e. the influence of the aluminum flakes in the oil can be neglected; the influence of gravity can be neglected;

the flow is isothermal, i.e. internal heating of the fluid because of dissipation can be neglected; viscosity and density are constant.

Imperfections are for instance:

the cylinders are not perfectly concentric;

a cylinder was not held rigidly enough and could perform small lateral movements;

the cooling device works reliably only up to certain Reynolds numbers.

To keep the effects both of idealizations and of imperfections reliably negligibly small sometimes required quite some efforts and inventiveness of the experimenters and is discussed in their papers. We shall mention further examples where they played an important role.

Figure 1.1 shows Taylor's apparatus as given in [Tay]. For technical details of later experiments see for instance [Co65, DS, BuKo, AC, ACDH, ALS, DL87, AH92], and the review article [Do92].

Following Taylor, comparison between experiment and mathematical model was always considered important. In some cases, the agreement between computed and measured values of flow parameters was excellent:

comparison of neutral curves of Couette flow by Taylor [Tay], see Fig. 1.4;

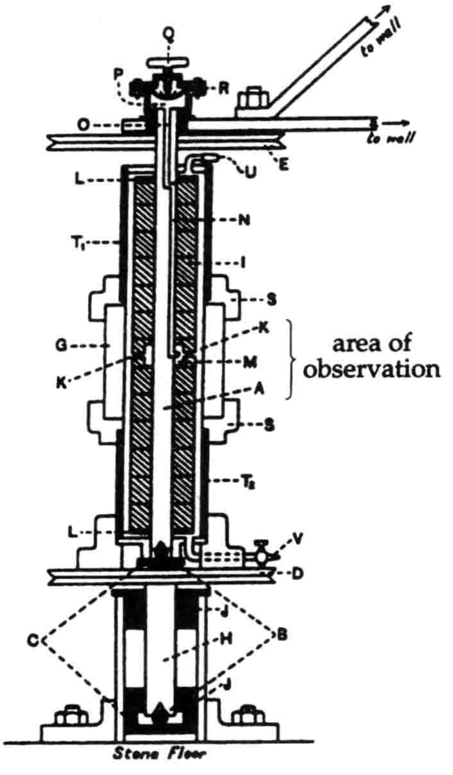

Figure 1.1: Taylor's apparatus [Tay, Fig. 8]. Observations were made in the range of the glass cylinder G.

comparison of torques for Re $\leq$ 2.2 Re_{cr} by Meyer-Spasche/Keller, see Figs. 1.2 and 1.3;
comparison of stability limits for Re $\leq$ 2 Re_{cr} by Riecke/Paap [RP86, Figs.2 and 3], see Fig. 3.37.

In several cases when agreement was poor, convincing explanations were found, and errors could then be reduced considerably. Examples:

- During the investigations of the Eckhaus instability leading to Fig. 3.37, it was found that gravity affected the results. After this was detected, its influence was excluded by using horizontal cylinders instead of vertically standing ones (see the section on the Eckhaus instability).

- In a case when computer simulation and experiment gave qualitatively the same, but quantitatively strongly differing results (the curves obtained were shifted from each other by a fixed amount), it was found that this was due to a systematic error in measuring [ClM85].

The only pronounced modeling disagreement in the Taylor vortex community is whether one can ever neglect the role of the aspect ratio of the experiment or if it always has to be taken into account [BM] (see subsection 'end effects' in this chapter and section 'validity of the numerical model' in the next chapter).

1.1.2 Mathematical description of the experiment

We assume that the experiments are performed with incompressible Newtonian fluids at constant temperature, with constant viscosity and constant density. We thus can describe them by the incompressible **Navier-Stokes equations** with no-slip boundary conditions at the two cylinders. The geometry of the apparatus suggests cylindrical coordinates. Let (r, θ, z) be the spatial coordinates, $\mathbf{v} = (u, v, w)$ the corresponding velocity components, p the pressure, ν the viscosity and ρ the density. The equations then read

$$\frac{du}{dt} - \frac{v^2}{r} = -\frac{\partial}{\partial r}\frac{p}{\rho} + \nu\left(\Delta u - \frac{u}{r^2} - \frac{2}{r^2}\frac{\partial v}{\partial \theta}\right) \tag{1.3}$$

$$\frac{dv}{dt} + \frac{uv}{r} = -\frac{1}{r}\frac{\partial}{\partial \theta}\frac{p}{\rho} + \nu\left(\Delta v - \frac{v}{r^2} + \frac{2}{r^2}\frac{\partial u}{\partial \theta}\right) \tag{1.4}$$

$$\frac{dw}{dt} = -\frac{\partial}{\partial z}\frac{p}{\rho} + \nu \Delta w \tag{1.5}$$

$$0 = \frac{\partial u}{\partial r} + \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z}. \tag{1.6}$$

Here

$$\frac{d}{dt}\cdot = \frac{\partial}{\partial t}\cdot \; + u\frac{\partial}{\partial r}\cdot \; + \frac{v}{r}\frac{\partial}{\partial \theta}\cdot \; + w\frac{\partial}{\partial z}\cdot \tag{1.7}$$

is the material derivative in cylindrical coordinates, and

$$\Delta \cdot = \frac{\partial}{\partial r^2}\cdot \; + \frac{1}{r}\frac{\partial}{\partial r}\cdot \; + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\cdot + \frac{\partial^2}{\partial z^2}\cdot\,. \tag{1.8}$$

There are several ways to *non-dimensionalize* (1.3)–(1.6): we can choose R_1, R_2, $\frac{R_1+R_2}{2}$ or $R_2 - R_1$ as characteristic length $\tilde{L}$, and we can choose $R_1\Omega_1$, $R_2\Omega_2$, $\frac{R_1+R_2}{2}(\Omega_2 - \Omega_1)$, ... as characteristic velocity V. With each of these